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Modelling Issues in Efficient Term Structure Estimation: A Study of the Indian Government Securities Markets

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Abstract

Estimation of the term structure of interest rates has been a subject of study for a long time now. Most of the studies have focused on the model specification that gives the least error defined as the difference between the model price/yield and the market price/yield. The Mean Absolute Error has commonly been taken as a measure of goodness of fit. In this paper, we study the modelling issues that would influence the performance of the parsimonious specification as given by Nelson Siegel and Svensson (1989) while constructing a zero coupon yield curve. These issues relate to the choice between efficient optimization algorithms used to solve the non-linear specification, the impact of starting parameters defined in the optimization, the influence of the objective function specified and, the input price considered for estimating the curve. These issues are seen to have some influence in a market that is not sufficiently liquid over all the maturity segments.

JEL Classification: E43, G1, G12, G15

Keywords: Term Structure of Interest rate estimation, Zero Coupon Yield Curve, Nelson Siegel Model, Bond Pricing, Optimization

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1 INTRODUCTION

The Government securities market is one of the critical components of any sovereign financial system. It is the primary borrowing avenue for the government and provides the benchmark to price other instruments in the debt market. It has also evolved to become the transmission channel for indirect instruments of monetary policy. The term structure of interest rates viz. the relationship between the yield and the term to maturity for the government securities market provides useful information about future movement in inflation, growth rate of the economy and is a mirror of market expectations. It is used by market participants for valuation purposes and the regulators for forecasting purposes.

The Indian Government securities market is characterised by trading activity in certain key maturity tenors across the term structure, which warrants the need to adopt efficient term structure estimation models that can accurately fit these traded tenor points. The fitting of the term structure needs to take into consideration factors like accuracy, ease of implementation and reliability.

Literature on term structure estimation has broadly developed into two directions- use of parametric and use of non-parametric models. While non-parametric models have looked at evolution of the long term rate from the short term rate, the parametric models aim to use multiple specifications for curve fitting across the term structure.

The commonly used term structure estimation models are bootstrapping method, polynomial/exponential spline methods and the exponential functional form methods of Nelson and Siegel (1987) and Svensson (1994). While bootstrapping is easy to implement, it does not perform optimisation and therefore as it fits each zero coupon yield to the bond prices, it may not produce a smooth curve. Cubic spline overcomes this shortcoming by segmenting the term structure over a series of points and fixing constraints to the curve at these points, thus ensuring the continuity and smoothness of the fitted curves. However, cubic splines may result in overfitting and give unreasonably curved shapes for the term structure at the long end.

The Nelson Siegel (NS) is a parametric model that uses exponential functions over the entire maturity range. The parameters are generally estimated by minimizing the sum of squared errors between the traded and model yields, subject to specific constraints, using an optimisation technique. The parameters can be interpreted as the long term rate, the slope, the curvature of the term structure. The Svensson model further refines the NS model by adding an additional flexibility which allows for broader and more complicated range of term structure shapes.

Most of the literature on term structure estimation has focused on the model specification. In this paper, the authors use the Nelson Siegel Svensson (NSS) specification to estimate the term structure for the Indian government securities market. The paper focuses on the modelling issues to be considered at the time of implementation, which would influence the performance of the NSS model. These include:

- **The Selection of the Optimisation Algorithm:** This study examines the fit of the NSS model under alternative optimisation algorithms. The choice between a constrained versus an unconstrained optimisation technique and its impact on the parameter stability is explored.
- **The Choice of Starting Parameter Values:** There has been little guidance in term structure literature on the ideal value of the starting parameters used to initialise an optimisation technique for yield curve estimation. This study puts in place alternative methods to determine the starting parameter values at the time of implementing the NSS model.
- **The Specification of the Objective Function:** The results of the yield curve estimation are also compared by way of implementing various objective function specifications. This includes minimising the sum of squared errors between the traded and model yields by assigning equal weights to all the errors. Alternative weighting schemes for the objective function by taking into account the size as well as the distribution of the errors are also explored.
- **The Input Price Specification:** The choice of input data is a key driver of a well specified model. The results of the yield curve estimation are compared by selecting a price based on entire days' trades, the last 3 trades only, and the last hour trades.

The yield curve was estimated under the various specifications of the objective function, the optimization, the starting parameter selection and the choice of the input price; to zero-in on a combination that would provide a best fit (viz. the model with the least objective function value), as well as parameter stability while implementing the NSS model.

Hence, the objective of this paper is two-fold:

1. To examine the modelling issues that would influence the performance of the parsimonious specification as given by NSS model while constructing a zero coupon yield curve. These issues relate to the choice between efficient optimization algorithms used to solve the non-linear specification, the impact of starting parameters defined in the optimization, the influence of that objective function specified and the input price considered for estimating the curve.
2. To create a historical database of the term structure of interest rates and the NSS parameters for the Indian government securities markets that can be used for macro and micro economic analysis.

This paper is organised as follows: Section 2 covers the published literature on the term-structure. Section 3 describes the data and filtering criteria used. The estimation methodology and micro-structure specifications adopted while implementing the NSS model are described in Section 4. The key results are given in Section 5. A historical database of the yield curve using the best specification of the NSS model for the Indian government securities market is provided in Section 6. Finally, Section 7 concludes.

2 LITERATURE SURVEY

There is a large body of literature on the specification of the best model to estimate the Zero Coupon Yield Curve (ZCYC). The two broad lines of thought include the parametric (splines) and non-parametric specifications (one-factor stochastic interest rate evolution)¹. Parametric models are defined by mathematical specification of splines which further categorized into two types the piece wise linear²/non-linear spline methods versus the parametric exponential methods as given by Nelson Siegel (1987) and Svensson (1994).

Nelson and Siegel (1987) introduced a parametrically parsimonious model for yield curves that had the ability to represent the shapes generally associated with yield curves such as monotonic, humped, and S shaped. The authors noted the model explained a significant variation in bill yields across maturities during the period 1981-83. The model was also able to predict the price of the long-term Treasury bonds suggesting that the model captured important attributes of the yield/ maturity relation.

Svensson (1994), estimates the forward interest rates with an additional hump parameter over the Nelson Siegel (1987) model and is commonly known as Nelson Siegel Svensson model. The study is done for Sweden for period of 1992-94. The paper compares the NS and NSS for the two years comparing price error minimization with yield error minimization. The paper concludes that while the additional flexibility of the yield curve does improve the results there are days when the simple NS model does better. The objective function when specified as price error minimization was seen to have an adverse impact on the fit at the short end.

For the Canadian Government bonds Market *Bolder and Stréliski (1999)* compared three yield curve models, namely Super-Bell, Nelson-Siegel (NS), and Nelson-Siegel Svensson Models (NSS). The study mainly deals with two key problems faced while constructing the yield curve. Firstly, the estimation problem, which implies choosing the best yield curve model specification by using multiple optimization techniques and alternative objective functions. Secondly, the paper considers the data problem, which involves selection of an

¹ Cox, Ingresoll, Rox (1985) family of models

² McCulloch (1971, 1975), Vasicek and Fong (1982), Darbha, G., Roy S., and Pawaskar V. (2003a)

appropriate set of market data. Under this study a simple OLS is used for Super-Bell Model, while for NS and NSS Models a log-likelihood specification of objective function and sum of squared errors are used to minimize price errors. The authors note that the objective function, which involves minimize price errors, can result in overfitting of long term bond prices relative to short-term bond prices. Hence, in order to correct this problem, the authors used an inverse of duration as weights associated with each price error. The study shows that the log-likelihood specification of objective function is an efficient approach while fitting the yield curve model. The authors used data filtering criteria, which filters the market bond data based on amount outstanding, term to maturity, divergence of YTM from coupon rate and inclusion of T-bills into the sample data. The study finds that using benchmark data filtering criteria to filter market data can lead to better results. Among all three models, the NSS model using a log-likelihood objective function and partial estimation algorithm was chosen as the best alternative by authors.

Subramanian (2001) studied the aspect of illiquidity in the GOI markets and the issue where yield curve estimation must include illiquid securities in the dataset. Pooling liquid and illiquid securities to estimate the term structure leads to errors in the estimation methodology. The author proposes the use of liquidity – weighted objective function for parameter estimation. The liquidity of individual securities is modelled using observable quantities like number and volume of trades in a security.

Dutta et al. (2005) also discuss about the case that in the presence of market frictions individual securities can be priced differently from the ‘average’ pricing of the market. Hence the model price errors are caused for two reasons (a) curve fitting and (b) presence of liquidity premium. The errors due to curve fitting arise from the calculations and should be avoided. But errors due to the presence of liquidity premium is reflective of market conditions and are to be addressed by using a weighted error function, with weights based on liquidity. They propose a liquidity weighted objective function where the liquidity function is specified using the exponential and hyperbolic tangent function to provide robust term structures in the Indian government securities markets.

Virmani, (2006) addresses issue with numerical optimization techniques for estimation and the sensitivity of results to the choice of initialization of the optimization routine. The study concludes that choice of starting values of the parameters is crucial for the solution obtained where the objective function value is at its minimum. The study shows that there exist regions in the shape of the objective function where a slight change in (seemingly reasonable) initial vector takes one far from optimum. The paper recommends a grid of reasonable starting values to be chosen and shape of the objective function value be assessed as the parameters are varied before narrowing down on a subset for which the objective function turns out to be at its minimum.

Garcia & Carvalho (2019) calibrated the NSS model for 20 countries, mostly Eurozone, to evaluate application of the NSS model under negative yield regimes. For forecasting purpose, the authors classified data into short, intermediate, and long-term maturities. The authors defined β_1 and $\beta_1+\beta_2$, parameters of the NSS model, as Interest Rate of a Very Distant Future (IRVDF) and Instantaneous Interest Rate (IIR), respectively. The study shows that difference between theoretical and observed values of the IIR is minimum as compare to the IRVDF. The authors find that the difference between theoretical and observed values of IIR and IRVDF, for all 20 countries, follows a normal and leptokurtic distribution, respectively. The study also observes that due to instability of monetary policy and the volatility of short-term interest rate the forecasting and fitting of yield curve model for short term data is difficult relative to intermediate and long-term data. The authors concluded that the NSS model is valuable tool to fit yield curves with negative yields.

Per Nymand-Andersen (2018) recommended the conceptual framework on extracting high-quality market data and applying statistical tests to evaluate and select an efficient yield curve model. The author used NS and NSS as parsimonious models, and Waggoner and Variable Roughness Penalty as class of cubic spline methods. The study uses Weighted Mean Absolute Errors, Root Mean Squared Errors and Hit-Rate as goodness of fit measures to evaluate yield curve models. The study shows that the NSS model provides a better goodness of fit under parsimonious class models, whereas Waggoner model showed an improvement in the results in case of cubic spline method. The author concluded that even though the spline based methods provide yield curve with slightly better goodness of fit as compare to the NSS model, the NSS model was better choice over all the other models as results produce by the NSS model were more transparent, communicative, and easy to interpret.

An estimation of the yield curve is a non-linear optimization problem. However, the issues which appear from optimization side are generally neglected and much focus is given towards various models which represent the yield curve. To address these issues the *Manousopoulos and Michalopoulos (2009)*, for Greek secondary bonds market, examine the NS and NSS models by using various optimization algorithms from group of Direct Search, Gradient Based, and Global optimization algorithms. The authors found that the direct search and global optimization algorithms outperform gradient based optimization algorithms in terms of average yield errors. The gradient-based optimization algorithms are generally dependent on starting points and they are restricted to find solution near the starting points whereas, direct search and global optimization algorithms explore large region of a search space. Thus, the studies suggests estimating yield curve using a direct search/ global optimization algorithms as a first step, and later refine the solution using gradient based optimization algorithm as a second step.

3 DATA

3.1 Data Description

The Indian bond market consists of Treasury bills (T-bills) issued by the central bank, Reserve Bank of India (RBI), through a pre-announced quarterly borrowing calendar. Treasury bills are issued for durations of 91, 182 and 364 days. Additionally, Treasury bills of any intermediate tenor, called Cash Management Bills (CMBs) are also issued by the RBI based on the extant liquidity requirements. Central government dated securities are issued by the RBI through auctions based on a pre-announced semi-annual borrowing calendar. The issuances of these securities are largely reissues of existing securities, with a few new issuances. Due to the expansion of the Indian economy, growth in number of participants and development of settlement and trading infrastructures, there has been a steady increase in the depth and volumes in this market (*Table 1*).

Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20
Turnover Ratio	121.46	107.87	110.72	181.21	214.17	219.81	187.40	309.36	184.67	137.61	174.31
Avg. Daily Volumes (Rs. Cr.)	12243	11623	14656	27353	37011	42853	40367	70017	47302	38657	54993

**Turnover ratio = Volume / Outstanding; Source: CCIL Calculations*

As of 2020 securities of maturity upto 40 years are outstanding in this market. In addition to fixed income securities, Floating Rate Bonds, Inflation Index Bonds and Special securities are the other securities which are currently outstanding. Trading in Treasury Bills and government securities takes place either on the anonymous electronic trading platform of NDS-OM³ or bilaterally with reporting on NDS-OM. Trading, especially in the recent period is largely concentrated specifically around the benchmark 10 year security (*Table 2*).

Financial Year	<= 3 years	>3 -7 years	>7-10 years	>10 - 15 years	>15 -20 years	> 20-25 years	> 25- 30 years	> 30 years
2009-10	9.93	30.96	34.31	18.66	3.95	0.82	1.37	0.00
2010-11	4.30	21.13	33.48	36.22	3.31	0.76	0.79	0.00
2011-12	1.18	8.83	54.81	31.67	2.27	0.27	0.95	0.02
2012-13	0.51	7.01	40.41	45.13	4.77	0.46	1.69	0.02
2013-14	1.08	13.57	43.18	37.99	2.98	0.14	1.05	0.01
2014-15	1.03	12.74	52.84	29.96	1.76	0.07	1.58	0.01
2015-16	2.59	15.98	52.05	25.37	2.00	0.17	1.82	0.04
2016-17	3.16	10.74	47.10	36.35	1.21	0.18	1.12	0.13
2017-18	3.50	10.63	46.08	36.64	1.92	0.21	0.80	0.23
2018-19	6.37	13.19	70.46	8.23	0.96	0.24	0.33	0.22
2019-20	9.06	19.89	61.95	7.27	0.44	0.63	0.36	0.40

Source: CCIL

³ Negotiated Dealing System – Order Matching – Anonymous trading platform for secondary market government securities transactions

Table 3 gives the maturity bucket wise turnover ratio. The ratio indicates that there is a concentration of trading in the benchmarks, generally in the 6 to 11 year bucket.

Table 3: Tenor Wise Turnover Ratio											
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20
0 to <3 Years	54.46	42.14	8.34	5.87	106.73	15.65	32.46	77.05	47.47	49.16	56.25
>=3 to <6 Year	69.87	43.71	9.02	39.51	83.38	102.15	111.75	115.18	74.73	104.22	188.51
>=6 to <9 Year	112.45	30.42	69.98	128.82	271.28	234.89	244.02	538.87	133.11	374.51	396.92
>=9 to <11 Year	393.44	277.25	479.91	300.46	2150.27	958.16	540.46	257.29	538.39	86.45	323.87
11+ Years	47.06	103.97	80.91	245.12	288.33	163.89	127.07	263.37	156.43	36.57	45.25
* Turnover ratio = Volume / Outstanding. Source: CCIL											

For the purpose of this study, securities traded during the period of 2009-10 to 2019-20 have been considered. The secondary market information on traded prices and yields, volume and number of trades is taken from NDS-OM. The 11-year period covers significant events that have had an impact on the Indian bond market, like the benign interest rate scenario of 2009 following the Global Financial Crisis, capital account crisis of 2013, the demonetisation period after November 2016, and beginning of the covid-19 pandemic.

3.2 Approaches to Data Filtering

The following data filtering criteria was adopted for the purpose of estimating the yield curve:

3.2.1 Security Selection

The starting point of the current study, which aims to develop the best fit model to estimate the yield curve, is security selection. T-bills, CMBs, along with dated government securities of maturity less than 1 year have been considered as short term securities. For the longer term, the goal is to use similar dated securities, with no special features that would affect their prices. In other words, the dated securities used in the study would ideally differ only in terms of their coupons and maturities. Therefore traded bonds with the following features are excluded from computation:

- All Floating Rate and Inflation Index bonds.
- Special securities like Oil Bonds, Fertilizer Bonds, and other bonds issued to specific institutions.
- All Principal and Coupon Strips.

3.2.2 Month-end Prices

As a precursor, the entire analysis was carried out on both a daily basis and monthly basis for the two year period from 2017-18 to 2019-20. It was found that the prices of the security as on the end of month were a good representation of the daily behavior of the security price through the month. Further, as the central bank norms for valuation of liquid securities in banks portfolio mandates valuation of such securities on a monthly basis,

there is also a possibility that the month-end data could have a higher quantum of traded securities, minimizing the possibility of in-sample errors. Hence the analysis was done on end-of-month basis for the entire sample period of 2009-10 -2019-20 for the purpose of evaluating each combination of the objective function, optimization, boundary condition and input price selection. On selecting the ideal combination, the historical NSS parameters were then estimated taking the daily price series for the full sample period of 2009-10 to 2019-20.

3.2.3 Securities with Standard Market Lot and Minimum 3 trades

The standard market lot for trading in the dated securities is Rs. 5 Crore (Rs. 50 million). There are trades that can happen in “odd-lots” i.e. lower than the lot size. A filter on securities is applied such that the trades of securities selected are of minimum Rs. 50 million. It is often challenging to filter purely on the basis of liquidity considerations. Hence, the data has been filtered to securities with a minimum of 3 trades on a given day. This basic filter is put in place to ensure that off-market prices of illiquid securities (securities with less than 3 trades) do not influence the optimization results. *Table 4* gives the number of securities traded before applying a minimum 3 trades filtering criteria. A comparison of the number of securities trade post the minimum 3 trades filter is presented in *Table 5*.

Table 4: Number of Securities Traded in each maturity bucket

Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20
0 to <3 Years	98	78	98	96	123	116	109	119	119	137	158
>=3 to <6 Year	13	16	8	12	12	12	16	17	17	19	19
>=6 to <9 Year	9	10	5	8	8	12	14	16	13	13	17
>=9 to <11 Year	4	5	5	6	6	5	9	9	11	9	10
11+ Years	17	11	11	15	20	21	26	31	27	25	28
Total	141	120	127	137	169	166	174	192	187	203	232

Table 5: Number of Securities Traded in each maturity bucket with 3 trades Filter

Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20
0 to <3 Years	54	35	43	49	67	66	60	71	64	73	89
>=3 to <6 Year	9	9	6	8	8	7	9	17	15	17	17
>=6 to <9 Year	8	6	3	5	8	8	11	14	13	12	14
>=9 to <11 Year	3	3	5	4	6	5	7	8	10	7	6
11+ Years	11	11	10	14	17	18	23	29	26	23	27
Total	85	64	67	80	106	104	110	139	128	132	153

3.2.4 Time Filter

The estimation of term structure of interest rates relies on the availability of traded prices at any given point of time during the day. This would require that the security is traded

frequently throughout the day with reasonable volumes. *Table 6* provides the average number of securities traded in hourly intervals throughout the day.

Table 6: Average Number of Securities Traded in Each Time Bucket											
Time interval (Hours)	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20
9AM-10AM	6	6	6	9	9	10	14	14	12	7	13
10AM-11AM	8	7	6	11	9	14	17	19	15	11	14
11AM-12PM	9	7	7	13	10	15	18	22	16	13	17
12PM-1PM	9	7	7	12	11	17	20	22	19	14	19
1PM-2PM	9	8	7	12	13	16	21	23	21	13	18
2PM-3PM	9	9	8	14	14	19	23	27	25	19	22
3PM-4PM	11	10	10	16	15	20	26	29	28	18	29
>4PM	13	12	11	16	17	22	31	34	33	25	34

The above table indicates that the trading is not uniform throughout the day. In addition, the price could also be influenced by news that would be available during the day. Hence it would be pertinent to consider alternative price inputs to arrive at an “appropriate price” for estimating the yield curve. Specifically, three price inputs are considered:

- The value weighted average price of all the trades in a security during the day
- The value weighted average price of the last 3 trades in a security during the day
- The value weighted average price of all the trade in a security during the last hour of trading.

4 METHODOLOGY

The term structure of interest rates has multiple applications and is used in areas such as valuing bond portfolio, determining appropriate trading strategies, capturing market risk, etc. Hence it is important to get a parsimonious model that would capture all the possible shapes of the yield curve. It is desirable that the implied forward rates are smooth and do not have any sharp kinks. The NS and the further improved NSS models have become popular due to the flexibility that they offer to determine the shape of the forward rates and hence the implied spot rates. The specification captures a monotonic form, humps at various areas of the curve and S-shapes. In this section the authors provide an overview of these two models and elaborate on the methodology adopted while estimating the ZCYC model.

4.1 Nelson Siegel Model

The NS model builds on the forward rate specification and integrates the same to get the spot rate. It ensures that smooth forward rates are derived on estimation. *Nelson Siegel (1987)* in their paper had focused on data that primarily encompassing Treasury Bills (securities dated less than a year) and uses only monthly data. The initial model

specification is on the forward rates which are integrated to give the interest rate function as a combination of three terms:

$$r(T) = \beta_0 + \frac{\beta_1(1 - e^{-T/\tau})}{\left(\frac{T}{\tau}\right)} + \beta_2 \left(\frac{(1 - e^{-T/\tau})}{\left(\frac{T}{\tau}\right)} - e^{-T/\tau} \right) \quad \dots(1)$$

β_0 , β_1 , β_2 , and τ are the constant parameters and T is the time to payment of the cash flow in annual units. The first term is constant, equals β_0 and defines the long-term level of zero rates, because the contribution of the other two terms diminishes as T approaches infinity. It remains constant for the entire term to maturity. The second term β_1 introduces an exponential time decay that becomes slower as τ increases. It has an impact on short maturities. The third term produces either a hump (if β_2 is positive) or a trough (if β_2 is negative) that occurs at a time governed by τ . Its impact increases with maturity, reaches a peak and then decays to zero.

4.2 Nelson Siegel Svensson Model

The NSS model was developed to provide additional flexibility to the NS curve. It adds an additional term to the existing Nelson Siegel specification with an additional local extremum along the maturity profile. The model for spot rate as derived by integrating the forward rate specification is given as:

$$r(T) = \beta_0 + \frac{\beta_1(1 - e^{-T/\tau_1})}{\left(\frac{T}{\tau_1}\right)} + \beta_2 \left(\frac{(1 - e^{-T/\tau_1})}{\left(\frac{T}{\tau_1}\right)} - e^{-T/\tau_1} \right) + \beta_3 \left(\frac{(1 - e^{-T/\tau_2})}{\left(\frac{T}{\tau_2}\right)} - e^{-T/\tau_2} \right) \quad \dots (2)$$

Where β_0 , β_1 , β_2 , β_3 , τ_1 and τ_2 are the constant parameters and T is the time to payment of the cash flow in annual units. The terms β_3 and τ_2 are the two additional parameters that provide an additional hump or a trough so as to provide flexibility of yield curve to be captured. This can be useful in improving the fit of yield curve.

The Nelson Siegel Svensson parsimonious models discussed here assume that the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. The spot rate is derived by integrating the forward rate function. The discount function is derived from the spot rates as:

$$\delta i = \exp(-r_t * t) \quad \dots(3)$$

Where, δi is the discount factor, r_t is the spot rate for a given time to maturity t .

The discount function is used to compute the model price of the bond (viz. spot price), given as the sum of discounted coupon payments, C_i , payable at time $i = 1$ to T and the present value of the redemption value to be paid at maturity T . The model price for bond b is therefore given as:

$$\text{Model Price}(b) = \sum_{i=1}^T C_i \delta_i + 100 \delta_T \quad \dots(4)$$

The model yield to maturity (YTM) is then estimated as an internal rate of return for the bond b that makes the present value of the coupon payments and the face value equal to the model price of that bond:

$$\text{Model Price}(b) = \sum_{i=1}^T \frac{C_i}{(1 + \text{Model YTM})^{t_i}} + \frac{100}{(1 + \text{Model YTM})^T} \quad \dots(5)$$

Where,

- $\text{Model Price}(b)$ is the spot price for bond b derived from the NSS model.
- C_i is the coupon payment payable at time $i = 1$ to T
- $\text{Model YTM}(b)$ is the model yield to maturity derived from the model price.

The parameters are estimated by minimizing the objective function, viz. the sum of squared errors between the traded and model price or yields, subject to specific constraints, using an optimisation technique.

The Nelson Siegel (1987) paper fixed the value of Tau parameter, to allow for a linear equation and determines the best model based on the lowest MAE. The literature since then has evolved to allow for non-linear optimization by imposing appropriate constraints on the parameters. This estimation of the parameters is based on the standard methodology well defined in literature and involves the following steps:

- i. Fix the vector of starting parameters $[\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2]$ for the first iteration.
- ii. Determine the instantaneous forward rate, zero coupon spot rates and discount factor functions using these starting parameters.
- iii. Use the discount factors to determine the present value of the bond cash flows which are summed to give the model bond price.
- iv. Compute the model YTM for each bond b from model bond price.
- v. Calculate yield errors taking the squared difference between the theoretical YTM and traded yield YTM.
- vi. Use a numerical optimization procedure to minimize the objective function for constraints on the short and long term rate to determine the new vector of parameters.
- vii. Use the new vector of parameters and repeat steps (ii) to (vi) until the objective function is minimized.

4.3 Optimization Techniques

4.3.1 Constrained Optimization by Linear Approximation (COBYLA)

The COBYLA is a local search, derivative free constrained optimization method. It optimizes an actual constrained optimization problem by approximating with a linear programming problem. The iterative linear approximation of the optimal solution of optimization problem is based on the method of *Nelder and Mead (1965)*. It constructs a linear approximation of the objective function and constraints over simplex of $n+1$ points and optimizes these approximations in a trust region at each successive iteration. As the algorithm proceeds in direction of optimum solution, the trust region radius also gets modified accordingly. The COBYLA minimizes an objective function using inequality and equality constraints where it converts equality constraints in a form of inequality constraints.

4.3.2 Limited-memory Broyden–Fletcher–Goldfarb–Shanno (L-BFGS)

The L-BFGS is an unconstrained optimization algorithm in the class of Quasi-Newton methods which approximates the BFGS algorithm. It is called limited memory method because the algorithm uses only low rank approximation of the Hessian matrix inverse instead of the entire Hessian inverse. The L-BFGS algorithm at each iteration, is a line-search, that is, a search along a ray in the variables, with the search direction computed from the approximate Hessian inverse. The L-BFGS is gradient-based method where gradient of an objective function is calculated numerically if not specified. This algorithm is well suited for optimization problem with many variables.

4.4 Objective Function in Term Structure Estimation

The objective function can be defined either in the price space or the yield space as the sum of square of the price or yield errors⁴. In this study, the errors are defined as the difference between the market YTM and model YTM. The estimation is done using maximum likelihood.

$$Objective\ function = \sum_{b=1}^N (Model\ YTM(b) - Market\ YTM(b))^2 \quad \dots(6)$$

⁴ Studies have indicated that using price errors have results in large yield errors for bonds and bills with short maturities as the prices are very insensitive to yields at short maturities. Hence it may be better to choose the parameters that minimize yield errors. This is also in sync with the monetary policy analysis that focusses on interest rates rather than prices. (Svensson, 1994).

Where,

- *Model YTM (b)* is the yield to maturity derived from the model price of the bond *b*.
- *Market YTM(b)* is the yield to maturity derived from the market price of the bond *b*.

While formulating the objective function, the errors used in the estimation procedure can be assigned equal weights or a weighting scheme could be designed to account for certain characteristics of the securities used in the input sample such as their liquidity. Additionally, the objective function can also be modified to take into account the distribution of the errors.

4.4.1 Objective Function Using Liquidity Weights

As illiquid securities might exhibit greater price/yield errors as compared to the liquid securities (since the prices of illiquid securities might account for a liquidity premia), a failure to incorporate liquidity of the securities at the time of estimation, could result in a difficulty to disentangle this liquidity premia from the model errors due to curve fitting. Hence, assigning higher weights to the liquid securities vis-à-vis illiquid securities and using the same in the objective function is expected to improve the fit of the resultant yield curve.

While, the weights for liquidity of securities can be defined by the reciprocal of their bid-ask spreads, a continuous series of bid-ask spread data at a regular frequency might not be available, especially for illiquid securities. Hence, literature has suggested various weights based on volume and number of trades in the securities (Subramanian (2001), Dutta et.al. (2005)). As such, the weights can be defined as:

$$W_i = \frac{\left(\left(1 - e^{-\left(\frac{v_i}{v_{max}} \right)} \right) + \left(1 - e^{-\left(\frac{n_i}{n_{max}} \right)} \right) \right)}{\sum_{i=1}^N W_i} \quad \dots(7)$$

Where,

- v_i and n_i are defined as the value of trade and the number of trades for the i^{th} security.
- v_{max} and n_{max} are the maximum value and the maximum number of trades for all the securities traded for the i^{th} day, respectively.

Using the above specification, securities with a lower number of trades (value) would be assigned a lower weight as compared to the securities with a higher number of trades (value).

4.4.2 Objective Function Using M-estimation weights

The resultant yield curve estimates can behave poorly when the errors are not normally distributed, particularly when the errors are heavy-tailed. In such cases, instead of removing such influential observations from the estimation process, one can consider using a weighting scheme in the objective function criterion that assigns lower weights to errors at the tails. In this paper, the Huber's M-estimation technique is selected for defining the weights associated with the yield errors and is explained as follows:

Step 1: Obtain a vector of yield errors, $[e_1, e_2 \dots e_n]'$ and the NSS parameters, by minimizing the objective function (sum of squared yield errors), by assuming equal weights to all the errors.

Step 2: Estimate the weights using Huber's M-estimation based on the vector of yield errors, obtained in Step 1, by defining the following variables:

Table 7: Derivation of Huber's M-estimation Weights		
Sr. No.	Variable	Equation
1	Mean of Vector of Yield Errors	$\bar{e} = \frac{\sum_{i=1}^n e_i}{n}$
2	Mean Absolute Residual	$MAR = e_i - \bar{e} $
2	Median of MAR	$md = Median\left(\frac{MAR}{0.6745}\right)$
3	Tuning Constant	$k = 1.345 \cdot md$
4	Absolute Scaled Errors	$abse_i = \left \frac{(e_i - \bar{e})}{md} \right $
5	Huber's Weight Function	$w(e_i) = \begin{cases} 1 & \text{for } abse_i \leq k \\ \frac{k}{abse} & \text{for } abse_i > k \end{cases}$
6	Weighted Squared Error	$wse = \sum_{i=1}^n w_i^2 \cdot e_i^2$
7	Weighted Average Error	$wae = \frac{\sum_{i=1}^n w_i \cdot e_i}{\sum_i w}$
8	Tolerance limit	10^{-16}

The tuning constant is picked to give reasonably high efficiency, so, k is defined as 1.345σ (where σ is the standard deviation of the errors) which produce 95% efficiency when the errors are normal, and still offer protection against outliers. Usually a robust measure of spread is used in preference to the standard deviation, wherein a common approach is to take $\sigma = MAR/0.6745$, where MAR is the median absolute residual (see *Fox and Weisberg, 2013 and Ruckstuhl, 2016*).⁵

The optimal weights using Huber's M-estimation are arrived at by using an iterative process. For the first iterative process, the absolute scaled error ($abse_i$) is computed as the deviation of the sample errors from the simple average error \bar{e} (and scaled by md). For subsequent iterations, the $abse_i$ is computed as the deviation of the sample errors from the

⁵ In this study, the MAR is computed as the mean absolute deviation.

wae obtained in the previous iteration (and scaled by md). The iterative process would continue so long as the difference between the wse obtained from the current and that obtained previous iteration reaches the tolerance limit.

Step 3: Re-estimate the ZCYC yield curve by using the NSS parameters obtained in Step 1 as the starting parameters and the optimal weights obtained in Step 2 to minimize the weighted yield errors.

4.4.3 Lorentzian Specification of Errors in the Objective Function

The field of robust statistics is primarily concerned with estimation problems, in which the data contains gross errors, or outliers that do not conform to the statistical assumptions for the majority of the data (e.g., Gaussian noise) ⁶. The main goals of robust statistics are therefore to:

1. Describe the structure best fitting the bulk of the data.
2. Identify deviating data points (outliers).

In our context, a robust statistic function $\rho(\cdot)$ can be applied on the squared yield errors to improve the robustness of the results from the yield curve estimation. One such robust statistic function that can be considered is the Lorentzian error norm. The Lorentzian error norm can be expressed as:

$$\rho(z_i, \sigma) = \log \left(1 + \frac{1}{2} \left(\frac{z_i}{\sigma} \right)^2 \right) \quad \dots(8)$$

Where,

z is the difference between the market yield and model yield.

σ is a threshold value. Threshold value is defined as 1 in this study.

The Lorentzian error function is less sensitive to outliers, since the function increases less rapidly when the value of the squared error terms exceed beyond a threshold value. Having expressed the squared errors using the Lorentzian form, we attempt to minimize the objective function as indicated below:

$$\text{Lorentzian error objective function} = \sum_{i=1}^n \rho(z_i, \sigma) \quad \dots(8)$$

4.5 Choice of Starting Parameters and Boundary Conditions

The search algorithm used in the estimation is across a non-linear curve that uses the various non-linear optimization algorithms. There are certain choices that are needed to be made with respect to the starting parameter before implementing the optimization

⁶ Heeger (1998)

process. Additionally, reasonable bounds are generally put in place for these parameters, to avoid the slowing down of the search process. This paper, tests for the sensitivity of the optimization solution to the starting parameters, by applying three alternatives approaches for defining the starting parameter values and the associated boundary conditions for β_0 and β_1 :

- **Arbitrary starting parameters:** In this approach, the starting parameters for β_0 and β_1 are not linked to the historical behavior of interest rates, but are defined arbitrarily. Additionally, the associated boundary conditions are not linked to the starting parameter values.
- **Static starting parameters:** In this approach, the starting parameter of β_0 is estimated as the average of the long term interest rate over the last 6 month period. The starting parameter for β_1 is defined as the difference between the average value of the long term and that of the short term interest rate over the last 6 month period.⁷ The starting parameters once decided would be fixed for the next 6 months. In this method the boundary conditions for β_0 and β_1 are defined as +/- 3% from its starting parameter values.
- **Dynamic starting parameter:** In this approach, the starting parameter of β_0 is estimated as the 6 month rolling average of the long term interest rate observed the last 6 month period. Likewise, the starting parameter for β_1 is defined as difference between the 6 month rolling average value of the long term and that of the short term interest rate. The starting parameter values are revised on a monthly basis. The boundary conditions for β_0 and β_1 are defined as +/-3% from its starting parameter values.

4.6 Traded Price Input

Three alternative price inputs to arrive at an “appropriate price” for estimating the yield curve were considered:

- **Value weighted average traded price (VWAP) for the Full day:** This would represent the average traded price by giving greater weightage to securities with the maximum value for the given day. It provides a benchmark to the market players to move in or out of the asset with as small of a market impact as possible. The traders would try to buy below the VWAP or sell above it thus maintaining the average levels. The VWAP levels are also used to indicate the trend in the prices of the underlying security.

⁷ The daily long term interest rate is calculated as the value weighted average rate of dated GOI securities with a maturity greater than 9 years. The daily short term interest rate is calculated as the value weighted average rate of all treasury bills, cash management bills and dated GOI securities with a maturity of less than and equal to 1 year.

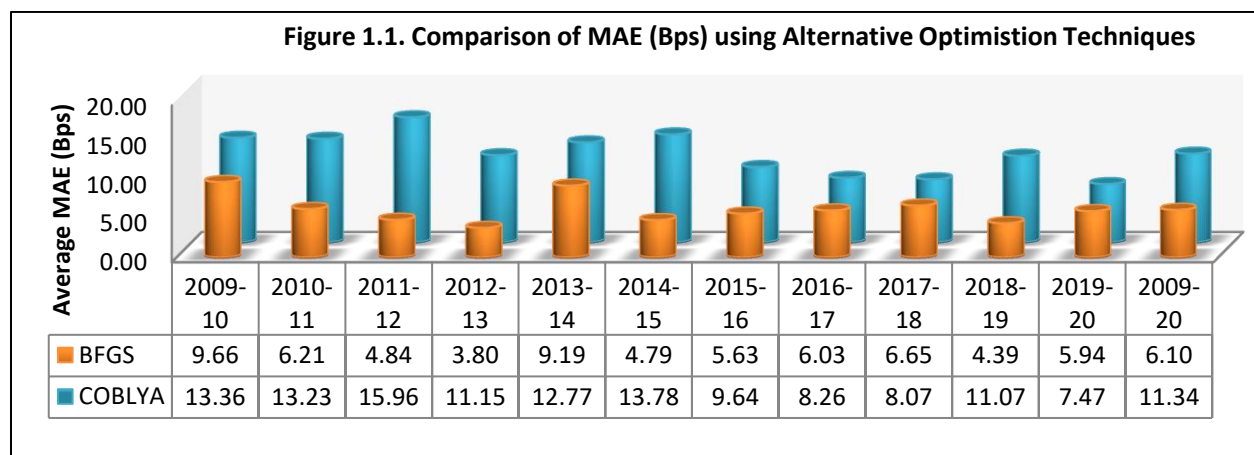
- VVAP for the Last Trading Hour: This price captures the latest information context just before the market closes and serves as a useful reference price on days when there have been significant changes in prices since the beginning of the trading session.
- VVAP for Last 3 Traded Prices: This price represents the value weighted average of the last three trades.

5 EMPIRICAL RESULTS:

Based on the methodology specified above, the NSS model was estimated on a month end basis from 2009-10 to 2019-20. In this section, the results are provided keeping in view that stability of parameters and the yield errors over time are the major considerations for evaluating the goodness-of-fit of the model. The goodness-of-fit of the models is captured by the mean absolute yield error and the standard deviation of the long rate and that of the short rate⁸.

5.1 Constrained Versus Unconstrained Optimization Methods:

Figure 1.1 compare the results of the NSS model in terms of the Mean Absolute Yield Error (MAE), Mean Absolute Price Error (MAPE) and the standard deviation of the long term rate (β_0) and short term rate ($\beta_0 + \beta_1$), when implemented using the constrained optimizer of COBLYA vis-à-vis the unconstrained optimizer of L-BFGS. Each of these optimizers were initialed with arbitrary starting parameters such that the starting parameters and associated upper and lower bounds were not linked to historical interest rate behavior. The results suggest that the average MAE were lower when implementing the NSS model using BFGS versus COBLYA.



⁸ A flowchart of the results under various combinations considered is provided in Annexure 1.

These results were supported by the tenor-wise analysis of MAE for the period in consideration under each alternative optimization technique (*Table 8*)⁹.

Table 8: Tenor-Wise Comparison of MAE (Bps) using Alternative Optimization Methods												
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: COBYLA Optimization												
0 To 5	9.60	11.94	14.79	11.46	16.04	18.19	10.25	6.93	5.12	10.74	7.78	11.26
5 To 10	16.01	12.58	12.86	9.44	14.20	12.56	9.35	8.88	9.22	10.50	8.70	11.54
10 To 15	12.46	8.18	8.02	5.11	6.96	9.52	5.45	8.96	10.38	8.67	5.66	8.27
15 To 20	26.43	21.28	23.71	15.50	7.00	13.53	10.11	7.93	8.67	15.42	5.60	14.00
20 To 40	25.01	21.15	27.73	22.18	9.04	20.18	15.00	10.80	10.42	17.51	7.13	16.29
Panel B: BFGS Optimization												
0 To 5	8.15	6.02	5.41	3.63	10.67	6.42	5.25	3.83	3.74	3.84	5.33	5.86
5 To 10	11.78	6.11	3.74	4.70	13.00	4.77	7.66	7.49	8.51	6.31	8.82	7.86
10 To 15	9.87	6.05	3.62	3.65	5.18	4.87	5.66	9.31	10.09	5.28	4.61	6.41
15 To 20	13.71	8.77	5.18	3.54	3.38	1.66	4.03	5.80	6.39	3.71	2.51	5.30
20 To 40	9.73	5.03	6.31	1.71	4.49	1.88	2.63	5.24	5.15	3.07	4.48	4.42

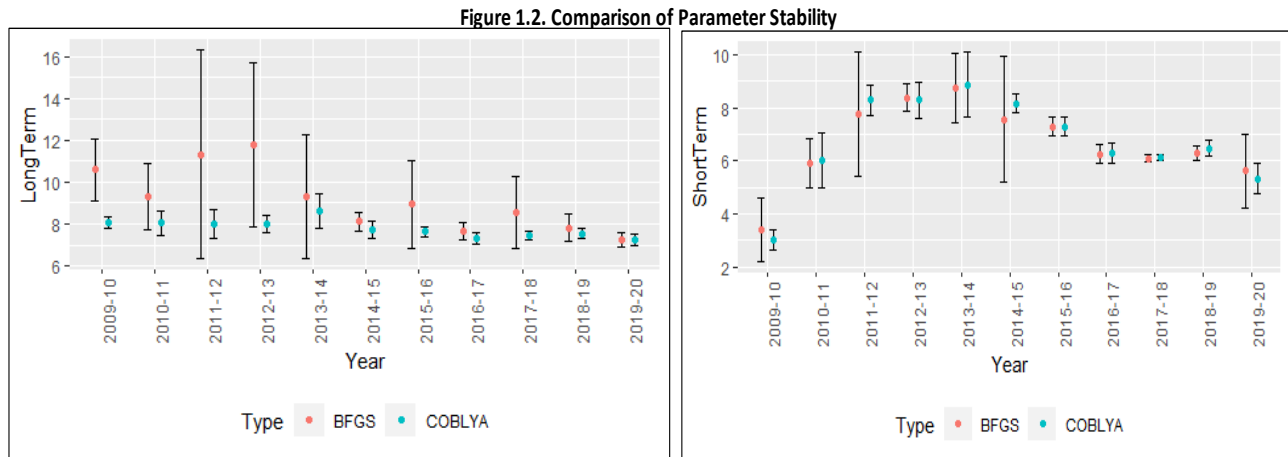
The hit rates analysis (*Table 9*), which shows the percentage of securities that fall under a certain basis point error thresholds, for the MAE computed using COBYLA and BFGS, concurred with the earlier results.

Table 9: Comparison of Hit Rates using Alternative Optimization Methods												
Threshold	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: COBYLA Optimization												
3Bps	19.21	12.47	15.34	17.37	16.85	14.77	17.96	20.41	25.93	15.94	28.56	18.44
5Bps	27.54	18.37	21.14	28.83	30.71	24.18	32.05	38.59	40.39	24.79	44.95	29.92
7Bps	38.24	27.79	30.51	44.16	42.55	33.42	43.35	55.29	52.90	36.93	61.58	42.18
10Bps	50.14	39.28	41.84	55.71	56.02	46.41	58.47	70.47	70.63	52.56	75.33	55.74
Panel B: BFGS Optimization												
3Bps	27.96	35.68	47.81	49.22	32.18	47.96	37.55	36.62	33.55	47.14	39.62	38.90
5Bps	39.48	50.55	68.71	74.12	46.14	67.10	59.03	55.60	49.34	66.70	58.02	56.88
7Bps	47.54	59.76	79.52	85.48	59.22	77.83	73.64	71.65	65.33	83.36	71.33	69.59
10Bps	62.28	76.89	88.43	95.77	71.62	87.59	86.20	82.71	80.48	91.33	82.62	81.61

However, it is pertinent to note that although BFGS seemed to provide a lower MAE as compared to COBYLA, it was observed that the parameters of the short term and the long term rates for BFGS proved to be unstable with a much higher volatility (+/-1 standard

⁹ A Comparison of MAPE is provided in Annexure 2.

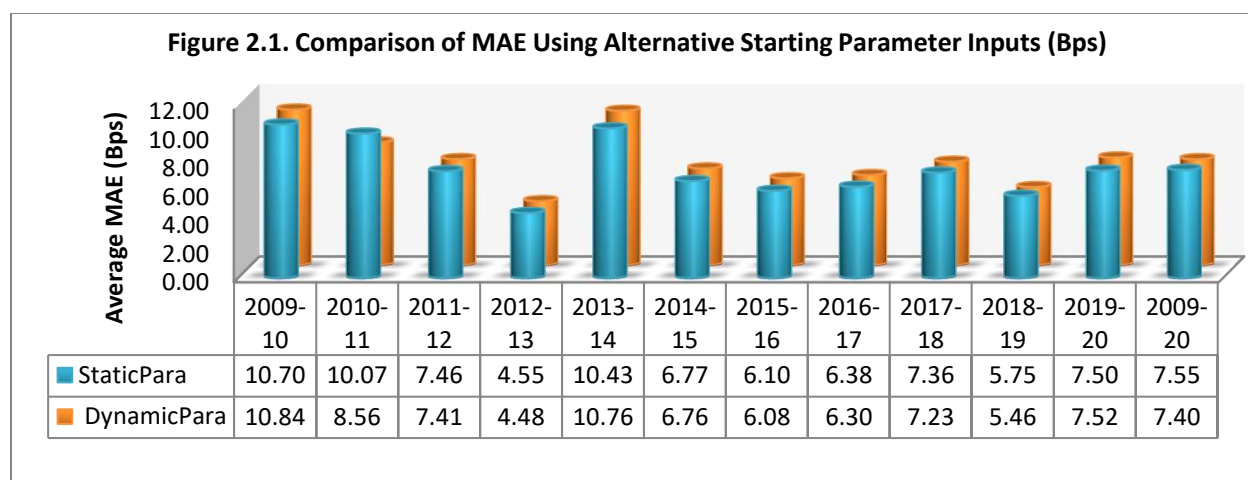
deviation). These results are highlighted in Figure 1.2¹⁰. It was also observed that, while introducing more stringent boundary conditions, the parameters in case of BFGS often hit the upper and/or lower bound, suggesting model instability, which was not the case for COBLYA. It was therefore decided to fine tune the ZCYC model estimation using a constrained optimizer such as COBLYA.



5.2 Choice of Starting Parameters

The estimation in Section 5.1 above, considered the computing the ZCYC using arbitrary starting parameters of β_0 and β_1 . To test the sensitivity of the optimization result to the value of the initial parameters used in the NSS model specification, two additional alternatives were considered, viz. static and dynamic starting parameters, such that the value of these parameters were linked to historical traded information. The MAE under each of these alternatives is provided in *Figure 2.1*. It was observed that linking the starting parameter values to the previous 6 month historical traded information seemed to provide better results, with a MAE of around 7.5 bps, which was much lower than the MAE of around 11 bps obtained from choosing arbitrary starting parameters while implementing a constrained optimization such as COBLYA.

¹⁰ Vertical lines in Figure 1.2 represent the upper and lower bounds using mean and ± 1 standard deviation, where, red and blue dots show mean parameter values for BFGS and COBLYA, respectively.



The results from choosing a 6-month rolling window for determining the starting parameters of β_0 and β_1 using traded information seemed to provide an edge over selecting a 6-month static window. The tenor-wise analysis of the MAE under each of these alternatives supported these results (*Table 10*).

Table 10: Tenor-Wise Comparison of MAE using COBLYA under Alternative Starting Parameter Estimates

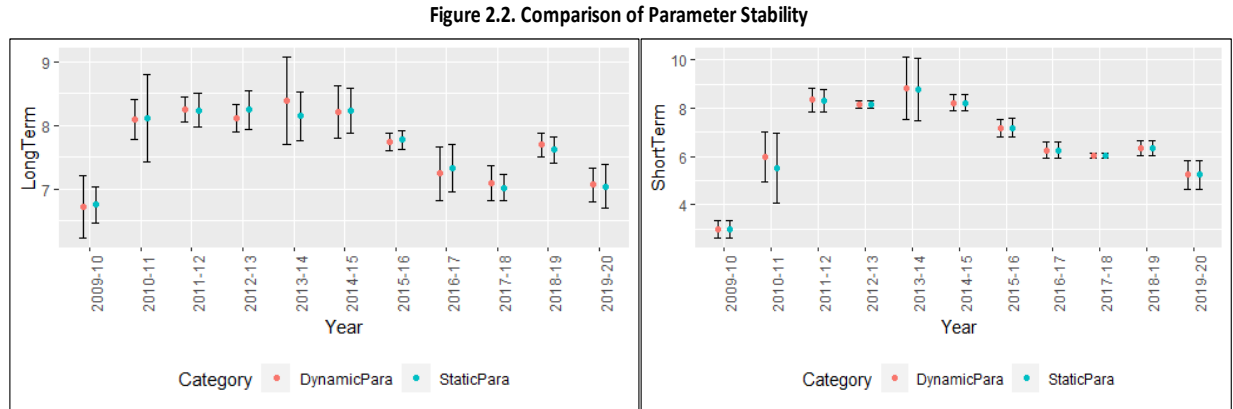
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Static Starting Parameters												
0 To 5	8.92	12.57	8.88	4.21	11.72	12.70	6.05	4.47	4.65	5.28	7.27	8.11
5 To 10	13.54	8.72	5.02	5.21	13.95	4.74	8.01	7.63	9.07	7.61	10.30	8.86
10 To 15	9.82	8.12	4.42	4.34	6.95	5.45	5.93	9.13	10.60	5.04	4.99	6.98
15 To 20	16.10	10.84	7.21	3.86	4.11	2.11	4.15	6.08	6.28	3.98	3.01	6.10
20 To 40	10.78	7.52	5.81	4.63	6.82	3.25	3.67	5.93	7.02	6.62	7.98	6.34
Panel B: Dynamic Starting Parameters												
0 To 5	9.43	10.02	8.80	4.40	13.00	12.60	6.05	4.49	4.59	5.10	7.33	8.01
5 To 10	13.82	7.60	4.72	5.08	13.10	4.71	8.17	7.48	8.93	7.09	10.46	8.59
10 To 15	9.81	7.40	4.60	4.14	6.96	5.56	5.93	9.21	10.51	5.63	4.69	6.93
15 To 20	15.67	9.17	7.46	3.83	4.80	2.43	3.76	6.01	6.20	4.00	3.19	6.00
20 To 40	11.41	6.65	5.05	3.99	7.95	2.72	3.05	5.50	6.23	5.03	8.16	5.90

A comparison of the hit rates under each of these alternatives is provided in *Table 11*, with around 75% of the total number of securities having a MAE less than 10 bps in each case.

Table 11: Comparison of Hit Rates using COBLYA under Alternative Starting Parameter Estimates

Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Static Starting Parameters												
3Bps	22.73	18.38	35.41	44.05	23.95	38.33	35.97	35.57	29.31	37.11	30.76	31.39
5 Bps	36.19	29.20	54.06	63.19	38.68	57.95	52.80	51.26	46.75	54.49	48.20	47.67
7Bps	48.39	43.56	68.24	79.32	49.61	68.97	68.35	68.13	60.63	68.57	60.78	61.48
10 Bps	57.07	59.93	81.02	90.51	64.92	82.76	84.28	81.68	72.71	85.61	72.29	74.94
Panel B: Dynamic Starting Parameters												
3Bps	19.10	26.43	36.71	42.12	23.52	38.65	33.45	35.42	30.15	37.62	30.38	31.60
5 Bps	36.14	37.23	54.89	64.19	38.49	57.67	52.74	51.14	46.74	59.77	48.43	49.04
7Bps	47.12	46.36	68.70	81.83	49.99	69.51	68.68	66.03	60.14	73.00	62.35	62.32
10 Bps	56.45	65.86	83.61	90.61	65.10	82.23	84.17	81.92	73.32	86.42	72.38	75.90

A similar behavior of parameter stability was observed in each of the alternative methods considered (*Figure 2.2*)



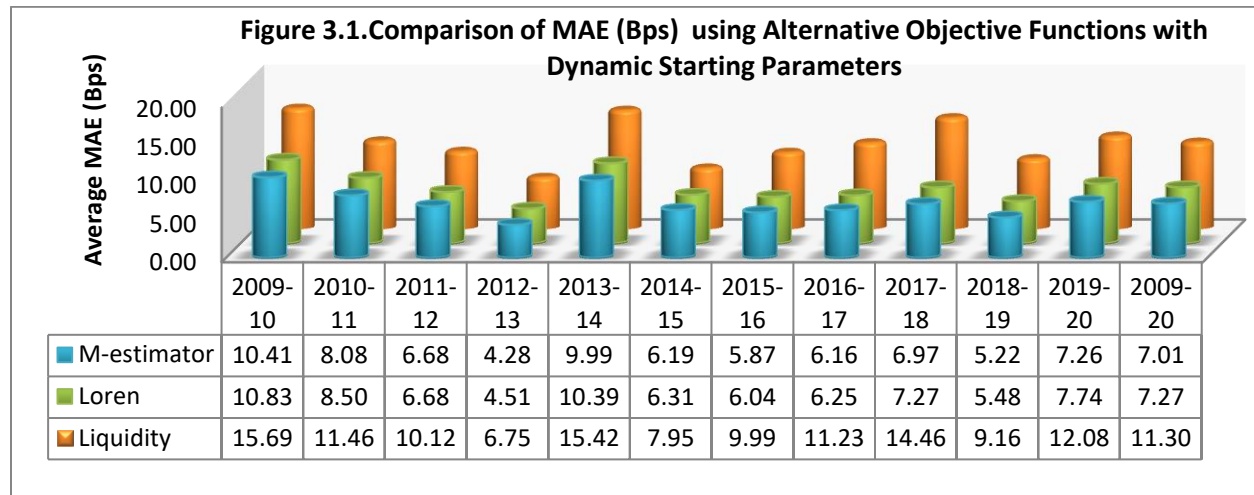
The constrained optimization technique of COBLYA using dynamic starting parameters was therefore selected for further analysis in the ZCYC estimation.

5.3 ZCYC Using Alternative Objective Functions

To improve the model fit, certain characteristics of the securities used in the input sample such as their liquidity were taken into consideration while formulating the objective function. The objective function was also modified to take into account the distribution behavior of the errors. The results are highlighted in *Figure 3.1*.

While following a liquidity-weighted scheme in the objective function did not seem to improve the results of the ZCYC model estimation¹¹, factoring the distribution of the yield errors seemed to provide better results. The MAE improved from around 7.50 bps when using an unweighted scheme to 7.27 bps and 7.01 bps when using the Lorentzian an M-estimation weighted objective function respectively.

¹¹ In fact, the MAE increased to around 11 bps using a liquidity weighted scheme versus 7.5 bps following an unweighted scheme in the objective function specification.



A comparison of the tenor wise MAE with the associated Hit Rates is provided in *Table 12* and *Table 13* respectively. Amongst each of the alternative specifications considered, the tenor-wise comparison suggests that the M-estimation based weights seemed to improve the results with a lower MAE as compared to an unweighted scheme.

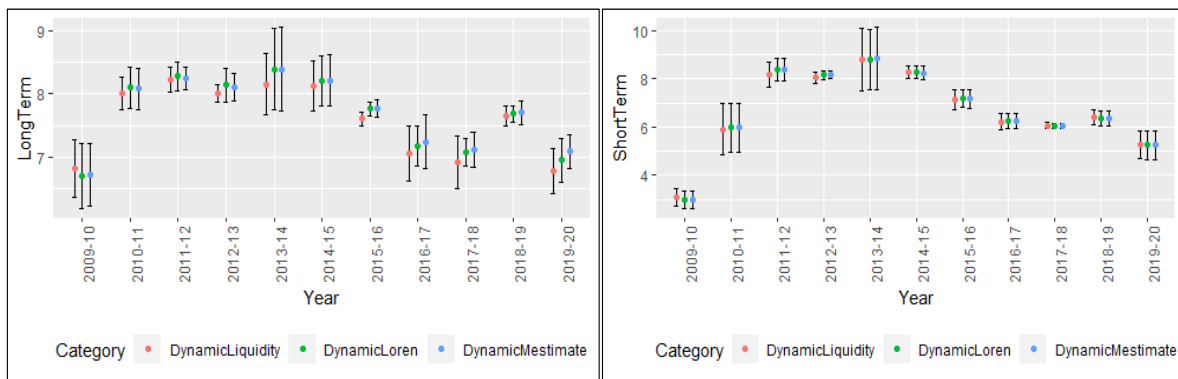
Table 12: Tenor-Wise Comparison of MAE (Bps) using COBYLA under Alternative Weighting Schemes (with Dynamic Para.)

Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Unweighted Errors												
0 To 5	9.60	11.94	14.79	11.46	16.04	18.19	10.25	6.93	5.12	10.74	7.78	11.26
5 To 10	16.01	12.58	12.86	9.44	14.20	12.56	9.35	8.88	9.22	10.50	8.70	11.54
10 To 15	12.46	8.18	8.02	5.11	6.96	9.52	5.45	8.96	10.38	8.67	5.66	8.27
15 To 20	26.43	21.28	23.71	15.50	7.00	13.53	10.11	7.93	8.67	15.42	5.60	14.00
20 To 40	25.01	21.15	27.73	22.18	9.04	20.18	15.00	10.80	10.42	17.51	7.13	16.29
Panel B: Liquidity Based Weighted Errors												
0 To 5	13.87	12.68	13.00	6.45	14.73	11.57	7.49	7.78	5.84	7.88	7.78	10.11
5 To 10	12.60	7.80	3.75	5.86	17.16	6.01	11.55	10.77	16.04	8.69	16.42	10.93
10 To 15	18.04	5.87	5.23	4.23	14.46	8.14	11.04	13.61	20.78	9.27	17.39	11.89
15 To 20	35.65	22.06	15.91	10.08	14.97	5.87	12.08	15.87	19.95	13.03	15.53	16.30
20 To 40	34.46	19.03	12.76	10.78	12.75	4.96	10.34	16.39	24.17	14.04	13.29	15.50
Panel C: Lorengian Weighted Errors												
0 To 5	9.48	9.90	7.46	4.32	12.08	11.15	5.91	4.33	4.64	5.15	7.45	7.65
5 To 10	13.88	7.49	4.99	5.33	13.59	4.76	8.21	7.48	8.96	7.05	10.73	8.70
10 To 15	9.84	7.41	4.37	4.28	6.84	5.42	5.91	9.25	10.45	5.70	4.95	6.92
15 To 20	14.91	9.24	7.66	3.83	4.37	2.10	3.82	6.03	6.21	4.03	3.13	5.89
20 To 40	10.91	6.86	4.12	4.14	7.79	2.51	2.95	5.41	6.49	4.72	9.33	5.87
Panel D: Huber's M-estimates Based Weighted Errors												
0 To 5	8.93	9.57	7.57	4.10	11.77	11.32	5.78	4.18	4.23	4.84	6.86	7.39
5 To 10	13.71	6.93	4.59	5.02	12.89	4.45	7.97	7.51	8.80	6.94	10.42	8.41
10 To 15	9.58	7.29	4.38	4.20	6.85	5.27	5.75	9.17	10.23	5.53	4.45	6.74
15 To 20	14.16	9.29	7.27	3.34	3.78	1.99	3.50	5.82	6.16	3.26	2.95	5.54
20 To 40	9.76	6.42	4.26	3.42	7.17	2.53	2.88	5.37	5.88	4.99	8.27	5.47

Table 13: Comparison of Hit Rates using COBLYA under Alternative Weighting Schemes (with Dynamic Para.)												
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Unweighted Errors												
3 Bps	19.21	12.47	15.34	17.37	16.85	14.77	17.96	20.41	25.93	15.94	28.56	18.44
5 Bps	27.54	18.37	21.14	28.83	30.71	24.18	32.05	38.59	40.39	24.79	44.95	29.92
7 Bps	38.24	27.79	30.51	44.16	42.55	33.42	43.35	55.29	52.90	36.93	61.58	42.18
10 Bps	50.14	39.28	41.84	55.71	56.02	46.41	58.47	70.47	70.63	52.56	75.33	55.74
Panel B: Liquidity Based Weighted Errors												
3 Bps	18.02	22.60	28.06	25.16	12.40	22.56	18.37	20.99	21.21	21.75	14.33	20.24
5 Bps	26.80	34.49	52.43	42.30	19.33	39.64	27.60	31.91	29.17	34.99	24.81	32.55
7 Bps	34.60	42.63	60.96	62.01	29.79	57.81	38.33	40.99	36.23	45.82	39.13	43.71
10 Bps	44.24	55.07	73.53	80.69	43.60	77.56	51.98	54.11	44.14	64.29	53.53	57.59
Panel C: Lorengian Weighted Errors												
3 Bps	19.09	24.21	40.38	44.92	25.16	38.94	33.02	35.14	29.96	38.82	28.65	32.06
5 Bps	34.36	38.15	56.64	66.86	37.28	58.75	53.56	53.58	45.68	58.08	46.76	49.27
7 Bps	47.24	45.48	68.48	79.31	51.17	69.82	69.75	67.01	60.22	73.46	61.29	62.35
10 Bps	56.66	67.07	84.64	91.87	66.60	83.87	82.84	82.18	73.03	85.35	71.97	76.29
Panel D: Huber's M-estimates Based Weighted Errors												
3 Bps	29.34	31.03	39.93	49.16	29.69	44.13	36.00	38.21	34.97	43.07	34.40	36.79
5 Bps	41.04	39.54	57.71	69.40	44.07	61.18	57.27	54.50	48.84	61.58	52.70	52.99
7 Bps	49.58	50.69	69.82	81.42	55.07	72.90	70.70	67.99	63.21	75.11	63.79	64.90
10 Bps	58.67	69.48	85.88	91.25	69.84	83.99	84.33	83.51	75.62	87.10	73.33	77.80

Figure 3.2 confirms the parameter stability of the Long term and short term rates under each alternative objective function estimated.

Figure 3.2 Comparison of Parameter Stability



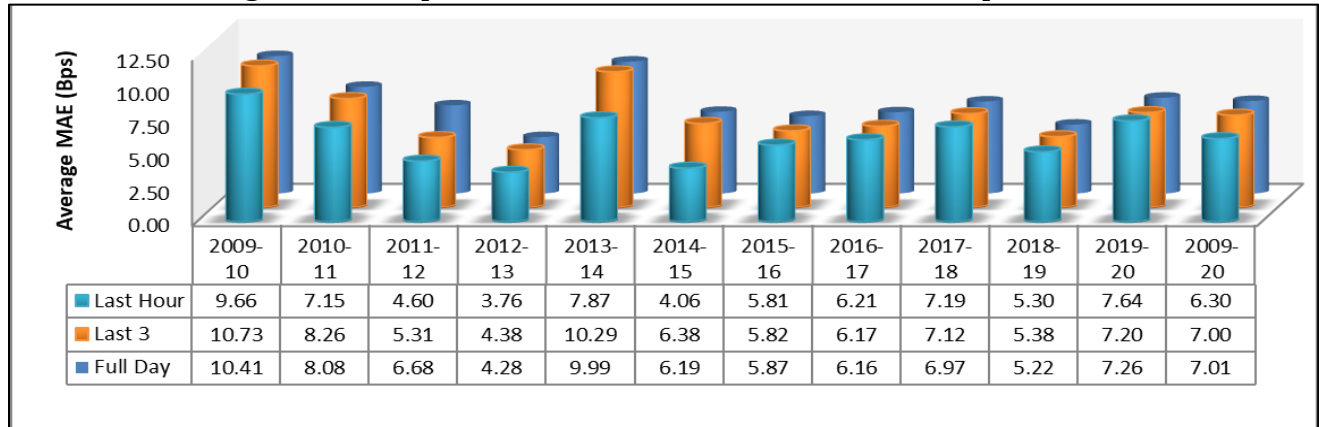
Based on the above analysis, the M-Estimation weight based objective function was selected for further fine tuning of the ZCYC model estimation.

5.4 ZCYC Using Alternative Price Inputs

Three alternatives were considered to test the impact of input price in the NSS model, the Full Day's Volume Weighted Average Price (VWAP), the VWAP of last three trades, on given

day is used as price input in model and the VWAP of securities traded in last hour for the day. *Figure 4.1* highlights that the MAE when using the price from the last hour of trading improved (6.30 bps) as compared to MAE derived from using the price for the full day (7.01 bps).

Figure 4.1 Comparison of MAE under Alternative Price inputs



The tenor wise analysis of the MAE concurred with these results (*Table 14*) with a marginal improvement in the hit rates (*Table 15*).

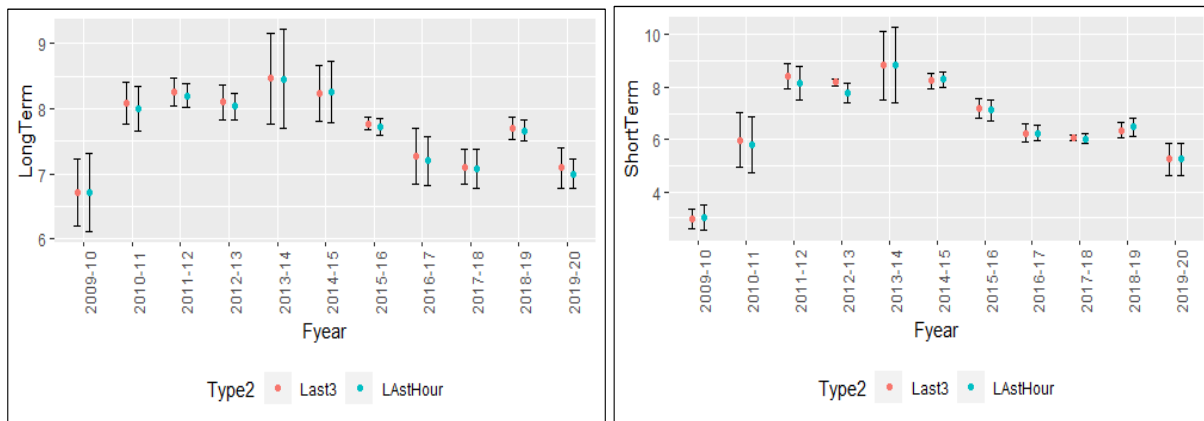
Table 14: Tenor-Wise Comparison of MAE (Bps) under Alternative Data Filtering Samples

Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Full Day's Trades												
0 To 5	8.93	9.57	7.57	4.10	11.77	11.32	5.78	4.18	4.23	4.84	6.86	7.39
5 To 10	13.71	6.93	4.59	5.02	12.89	4.45	7.97	7.51	8.80	6.94	10.42	8.41
10 To 15	9.58	7.29	4.38	4.20	6.85	5.27	5.75	9.17	10.23	5.53	4.45	6.74
15 To 20	14.16	9.29	7.27	3.34	3.78	1.99	3.50	5.82	6.16	3.26	2.95	5.54
20 To 40	9.76	6.42	4.26	3.42	7.17	2.53	2.88	5.37	5.88	4.99	8.27	5.47
Panel B: Last 3 Trades												
0 To 5	9.43	9.88	5.07	4.05	12.53	11.83	5.79	4.25	4.27	4.94	6.76	7.36
5 To 10	13.60	7.35	4.72	5.16	13.32	4.55	7.74	7.41	8.97	7.00	10.35	8.51
10 To 15	10.00	7.32	4.38	4.64	6.37	5.11	5.72	9.06	10.63	5.57	4.48	6.80
15 To 20	14.00	8.80	7.79	3.51	3.81	2.25	3.36	5.82	6.07	3.59	2.77	5.57
20 To 40	10.52	6.31	4.14	3.67	7.19	2.60	3.07	5.62	6.01	5.78	8.36	5.71
Panel C: Last Hour of Trading												
0 To 5	5.15	5.41	2.70	1.81	5.14	2.73	4.79	3.67	3.50	3.98	6.58	4.47
5 To 10	13.72	8.99	5.55	5.62	12.40	4.60	8.16	7.57	8.57	8.67	10.95	8.94
10 To 15	9.27	6.38	1.93	3.23	6.46	5.62	6.09	8.40	10.13	4.47	4.93	6.27
15 To 20	15.76	11.73	8.96	3.76	4.01	1.90	3.85	4.82	6.20	5.35	3.06	6.16
20 To 40	5.56	3.62	5.37	3.92	7.18	1.86	1.93	5.18	6.40	7.31	6.39	4.91

Table 15: Comparison of Hit Rates using COBLYA under Alternative Data Filtering Samples with Dynamic Para. And M-estimator												
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Full Day's Trades												
3 Bps	29.34	31.03	39.93	49.16	29.69	44.13	36.00	38.21	34.97	43.07	34.40	36.79
5 Bps	41.04	39.54	57.71	69.40	44.07	61.18	57.27	54.50	48.84	61.58	52.70	52.99
7Bps	49.58	50.69	69.82	81.42	55.07	72.90	70.70	67.99	63.21	75.11	63.79	64.90
10 Bps	58.67	69.48	85.88	91.25	69.84	83.99	84.33	83.51	75.62	87.10	73.33	77.80
Panel B: Last 3 Trades												
3 Bps	24.37	31.50	41.22	49.25	27.73	42.82	36.73	37.85	36.98	40.11	35.69	36.11
5 Bps	34.54	40.92	58.63	67.31	43.38	60.38	56.21	53.50	50.03	58.41	52.85	51.71
7 Bps	45.83	51.73	68.57	80.05	54.50	71.98	72.00	68.92	61.62	75.21	64.46	64.39
10 Bps	57.94	66.66	83.76	91.44	71.89	83.42	84.93	82.71	75.26	85.62	74.06	77.33
Panel C: Last Hour of Trading												
3 Bps	30.86	32.38	51.28	52.81	31.81	47.78	34.94	38.95	31.58	43.19	30.17	38.52
5 Bps	45.46	50.15	64.40	71.22	50.28	69.09	56.10	56.06	48.48	62.12	47.16	55.84
7 Bps	53.42	63.86	72.87	79.92	62.09	80.78	74.13	69.02	65.19	74.65	60.13	68.03
10 Bps	67.75	71.62	85.99	94.01	77.66	92.41	84.72	81.82	74.65	83.27	73.56	79.71

Notwithstanding, in each of the price input alternatives a similar behavior in parameter stability was observed. So as to capture the information content in the prices of the securities traded throughout the day, the VWAP of the entire day's rate was considered for the purpose of daily estimation of the ZCYC model.

Figure 4.2 Comparison of Parameter Stability



To summarize the findings, it was observed that a constrained optimization was better suited in the ZCYC yield curve estimation vis-à-vis an unconstrained optimization. It was further observed that the optimization solution is often sensitive to the starting parameters selected while implementing the NSS model and was found that linking the starting parameter values of β_0 and β_1 to historical traded information could improve the estimation results. When evaluating the effectiveness of the objective function, it was found that adopting a weighting scheme based on the distributional properties of the yield errors at the time of estimation helped in lowering the MAE on an average. The input price used

was also found to be an important consideration in the ZCYC model estimation in terms of a lower MAE. However, the resultant parameters were found to be stable irrespective of the price input used.

6 CREATING THE HISTORICAL DATABASE OF TERM STRUCTURE OF INTEREST RATES BASED ON NSS PARAMETERS

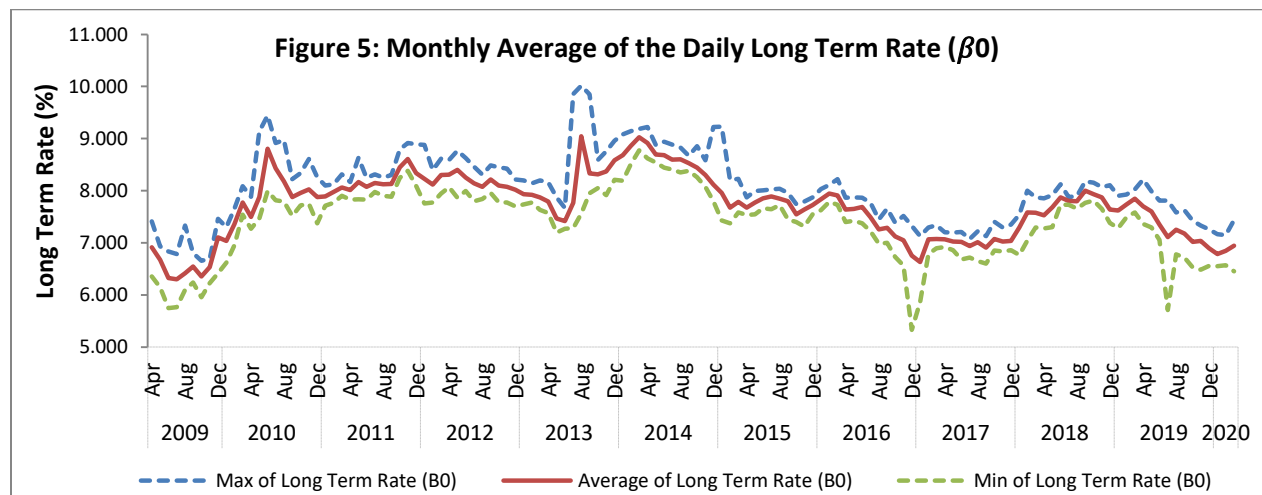
Based on the analysis in section 5 above, the following specifications were incorporated for estimating the ZCYC in case of the Indian government bond market securities for a historical daily sample period of 2009-10 to 2019-20:

- Use of a constrained optimization (COBYLA) technique.
- Use of Rolling Starting Parameter values for β_0 and β_1 based on 6-month historical traded information.
- Use of an objective function specification based on M-estimation-weighted errors to account for the distribution of the errors.
- Use of a Value Weighted Average Price of securities trades across the entire day.

6.1 NSS Parameter Stability

NSS model was implemented to arrive at the estimated values of the parameters based on the above criteria. The behavior of the long run parameter (β_0) and the short run parameter ($\beta_0 + \beta_1$) are provided in *Figure 5* and *Figure 6* respectively. The long term parameter was on an average in the range of 6.30%-9.05% while the short term rate was on an average in the range of 2.59%-10.61%, indicating a greater variation in the short end of the curve vis-à-vis the long end of the curve during the period in consideration.

A comparison of the monthly variation as depicted by the monthly maximum value (blue line) over the monthly minimum value (green line) suggests that the daily long term and short term parameters were found to be relatively stable over a given month. It is pertinent to note however that both the long term and short term parameters do evolve over time.



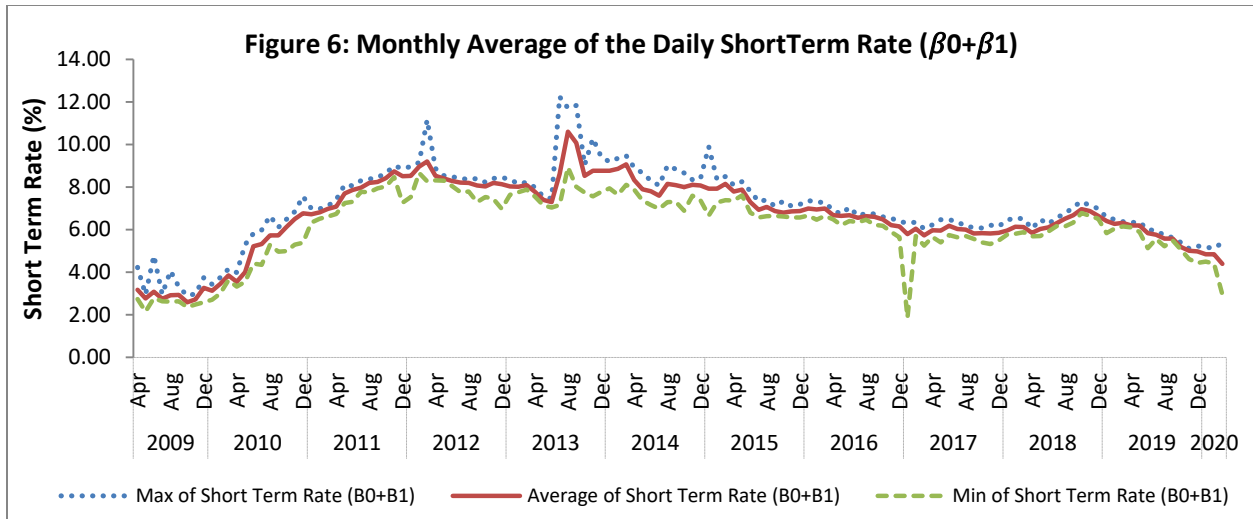
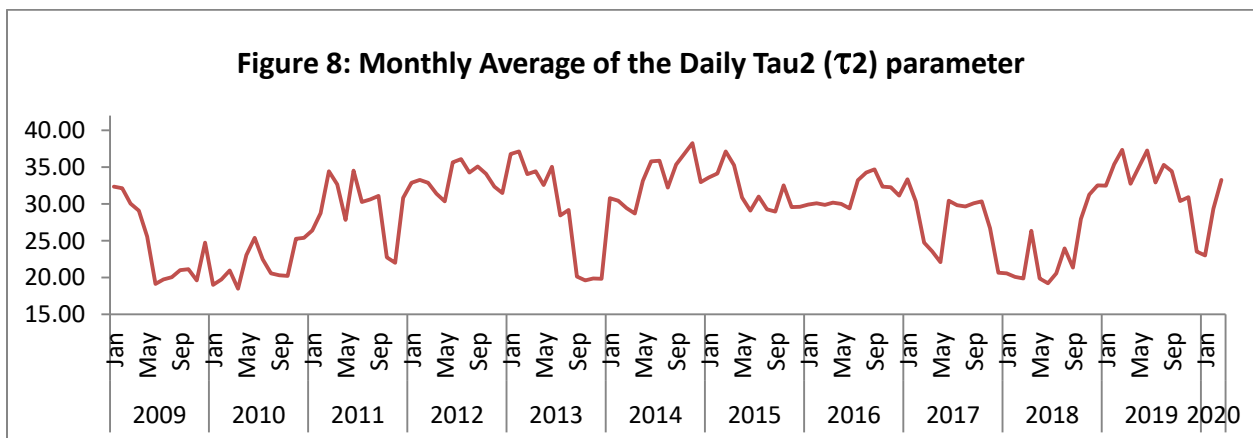
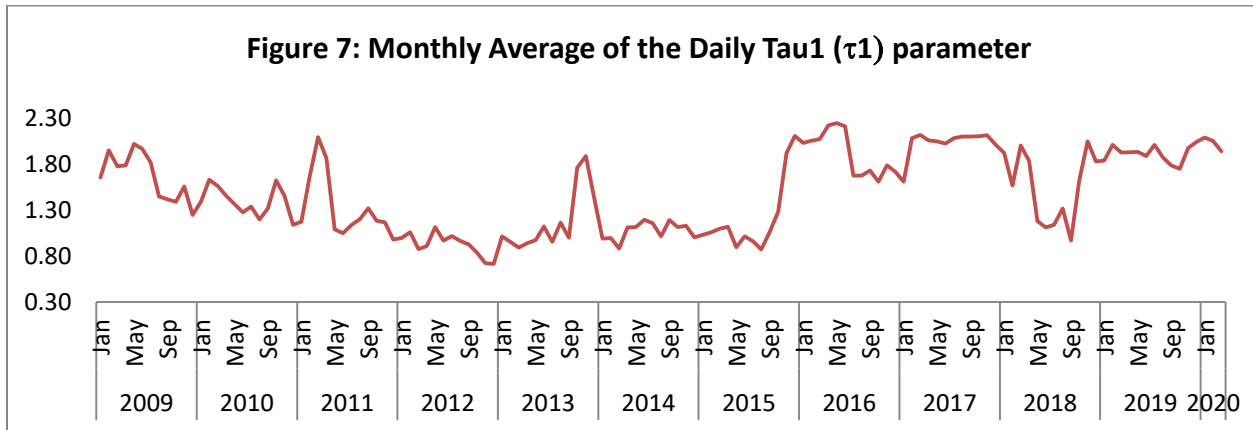
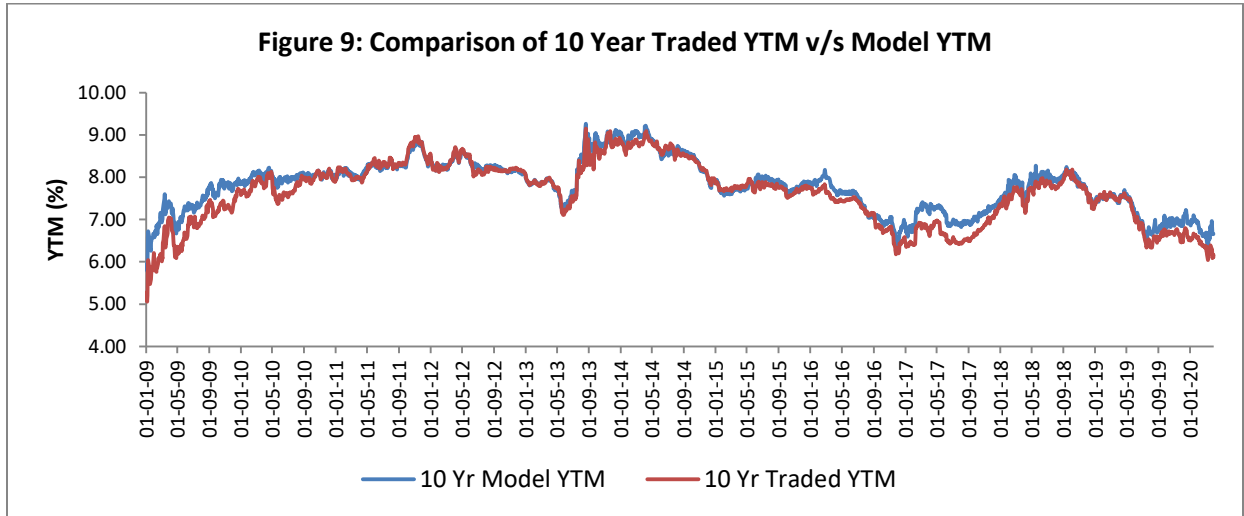


Figure 7 and Figure 8 represent stability of tau1 (τ_1) and tau2 (τ_2) parameters respectively. The tau parameters determine position of hump on the yield curve. Where, τ_1 shows first point of inflection at shorter end of the yield curve and τ_2 shows position of second hump at longer end on the yield curve. The τ_1 parameter was in the range of (0.72-2.22) and τ_2 parameter was in the range of (18.47-38.27).



6.2 Evolution of the Term Structure

The estimated NSS parameters were used to derive the model YTM curve for the period of 2009-10 to 2019-20. As a preliminary analysis, a comparison of the daily YTM of the traded 10 year government bond with the 10-year YTM implied from the NSS ZCYC curve was made (*Figure 9*). The results reveal that the NSS specification effectively modelled the 10 year traded rate.



The evolution of the term structure over the entire period of 2009-10 to 2019-20 is provided in *Figure 10*. The figure captures the different shapes of the YTM term structure over the period considered, such as upwards sloping, flat, inverted curve etc. The figure also captures the overall decline in the level of the curve.

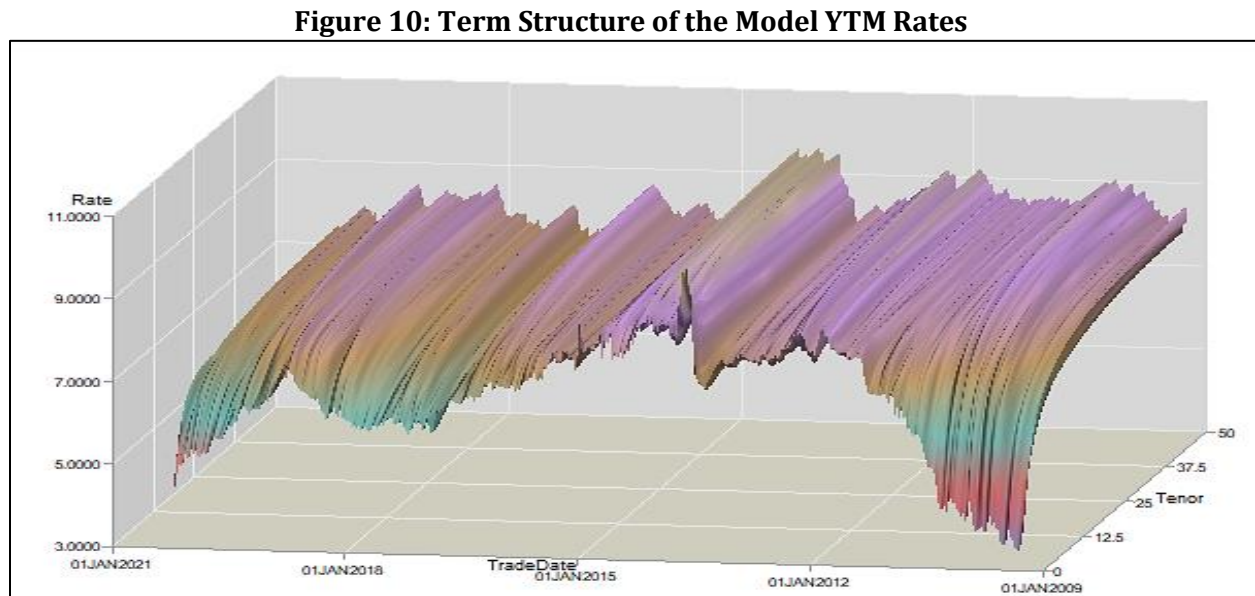
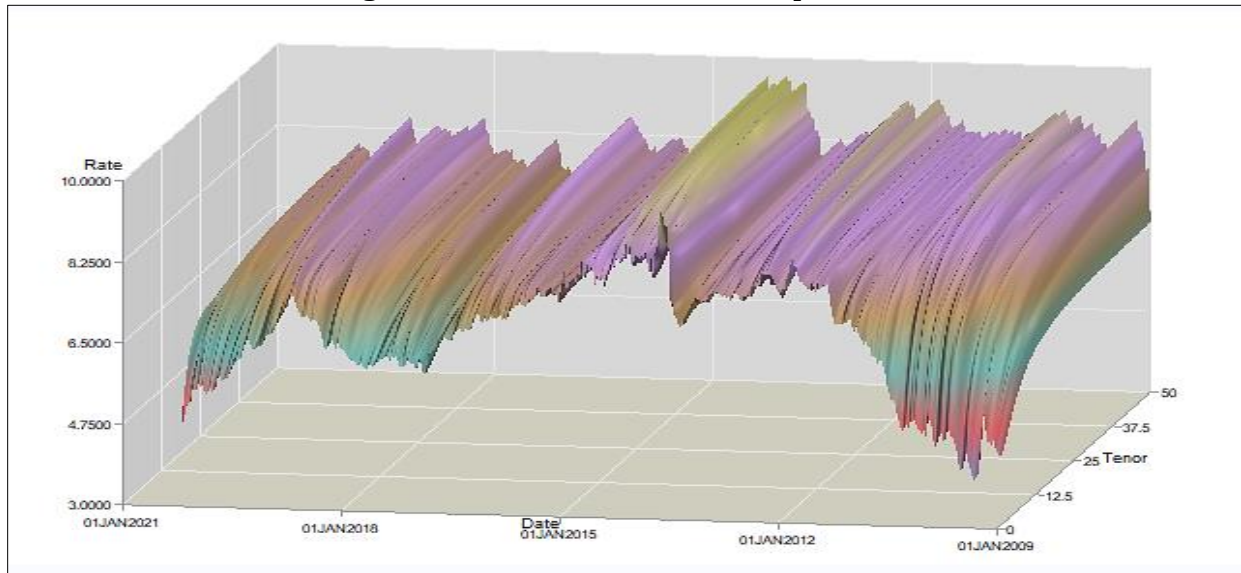


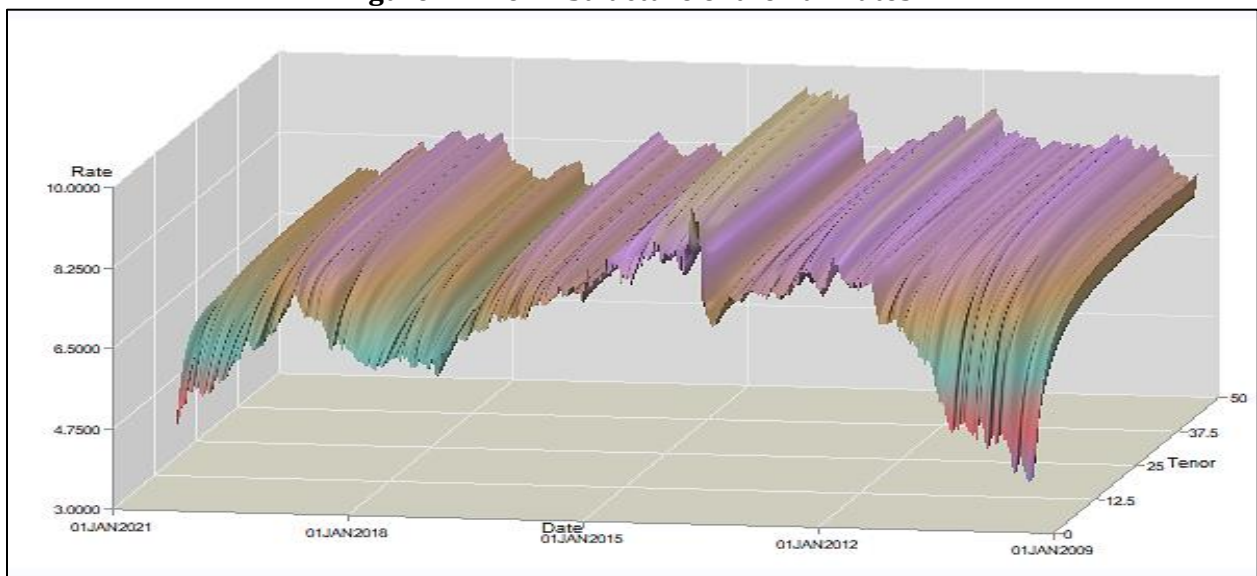
Figure 11 illustrates the term structure of the spot rates that was estimated from the NSS model and used for deriving the implied YTM curve. A spot rate represents the rate used to arrive at the discounted present value of a Rupee for a given maturity on the term structure.

Figure 11: Term Structure of the Spot Rates



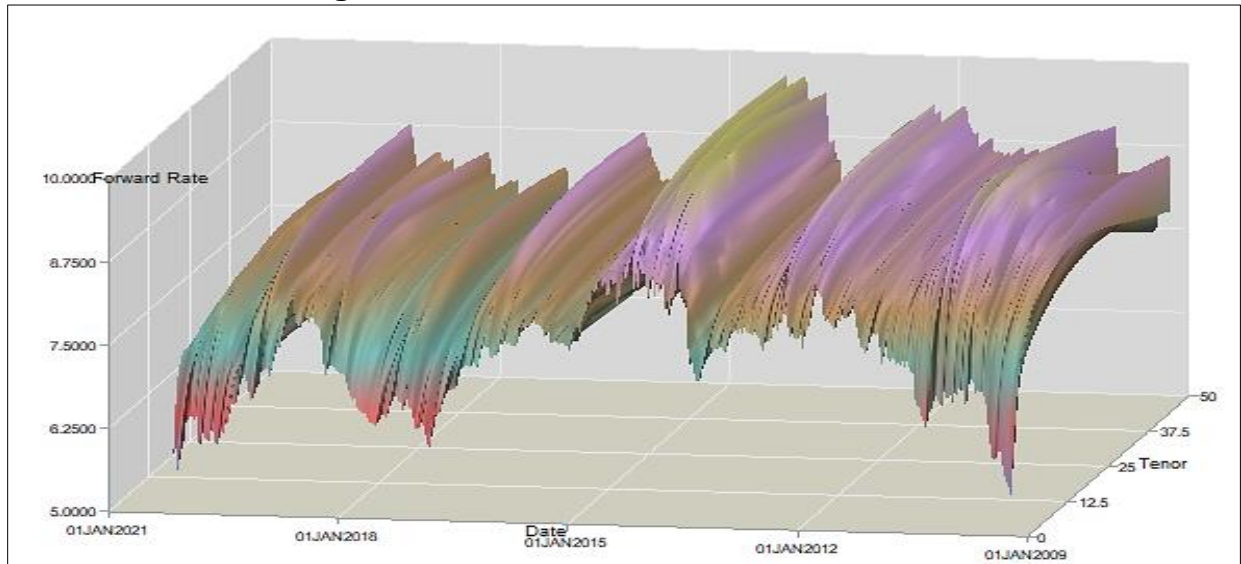
The evolution of the Par curve over the entire period is provided in Figure 12. The Par curve is a representation of the yield to maturity implied from the spot curve under the assumption that the bond is traded at par (i.e. the bond price is fixed at 100). The Par curve is a useful indicator for market participants to determine what the coupon of a newly issued bond could be for a specific maturity on the curve on any given day.

Figure 12: Term Structure of the Par Rates



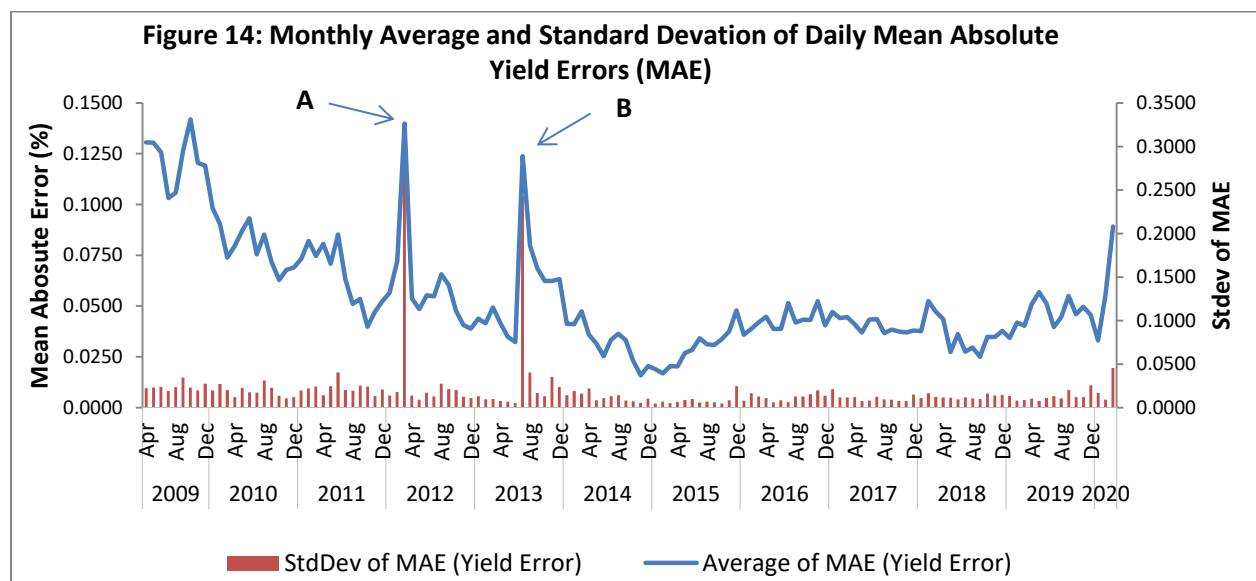
The yield curve is also often used as a tool for forecasting expected interest rates movements. For example, the spot rates of 1 month and 2 month can be used for deriving the 1 month implied forward rate a month later. Using the spot curve, the 1-month ahead forward curve was implied for the period in consideration. The results are illustrated in *Figure 13*.

Figure 13: Term Structure of the Forward Rates



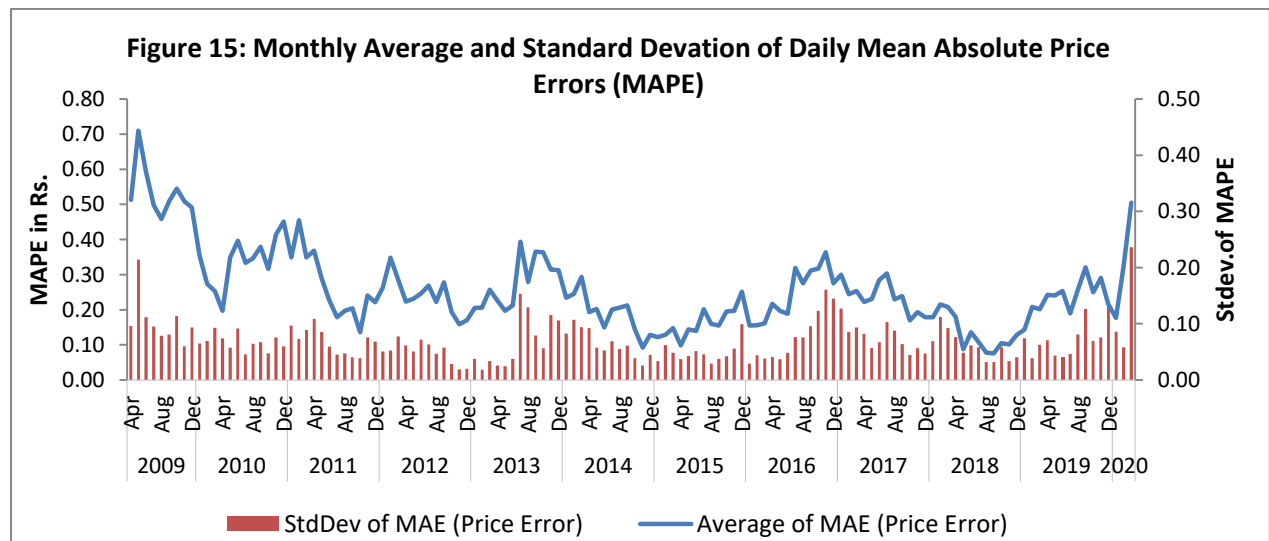
6.3 Price and Yield Error Analysis

Based on the estimated term structure, the model price/yield of the securities were estimated and compared with the traded price/yield. The monthly average of daily mean absolute yield errors and price errors over the entire period with the associated the standard deviations are provided in *Figure 14* and *Figure 15* respectively.



In the *Figure 14*, monthly average and standard deviation of MAE is high at point A and B. A higher MAE can be result of increased fluctuations in the government securities market or can be related to a particular security. A detailed analysis revealed that a spike at point A represents effect of inclusion of short term to maturity security in a data sample for particular day. The yield of short term to maturity security is highly sensitive to small price change. Hence calculating yield of such security from modelled price is result in larger error. Some literature suggests exclusion of such short term to maturity security from model estimation process.

The government security yields are generally impacted by various macro-economic factors. The point B in *Figure 14*, exhibits increased volatility in the government securities market, during May-Jun 2013, due to taper tantrum of US Federal Reserve. The Fed's taper tantrum altered global monetary and financial conditions dramatically with spillovers in the Indian markets. The RBI had reacted with tightened liquidity and increased interest rate in order to safeguard the domestic currency. Three specific monetary measures were announced by RBI to tightened liquidity conditions: a) Marginal Standing Facility (MSF) rate was fixed at 300bps, b) access to overnight liquidity from RBI was restricted to 1 percent of Net Demand and Time Liabilities (NDTL) of the banking system, and c) announcement of open market sales of 120 billion.



A tenor-wise comparison of the mean absolute price errors are provided in Table 16. The results suggest that the price errors on an average were around Rs.0.07 for securities less than 1 year, around Rs. 0.36 for securities between 5 years to 10 years and around Rs. 0.54 for greater than 15 years.

Year	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	2009-20
0 To 5 Yrs.	0.1446	0.1167	0.0455	0.0408	0.0951	0.0776	0.0674	0.0673	0.0784	0.0585	0.0816	0.0795
5 To 10 Yrs.	0.8265	0.4504	0.3590	0.3123	0.3569	0.1026	0.2259	0.2993	0.2970	0.2138	0.5149	0.3608
10 To 15 Yrs.	0.8654	0.6039	0.3136	0.4278	0.4258	0.1453	0.1403	0.5293	0.3607	0.2264	0.3069	0.3979
15 To 20 Yrs.	1.1335	0.8704	0.6940	0.2884	0.3261	0.2615	0.2759	0.5212	0.6520	0.3006	0.5490	0.5419
20 to 40 Yrs.	0.6225	0.5938	0.4552	0.3840	0.5497	0.4137	0.3105	0.8654	0.6864	0.4847	0.4965	0.5274

7 CONCLUSION

This paper uses the Nelson Siegel Svensson (NSS) specification to estimate the term structure for the Indian government securities market. The study addresses key modelling issues at the time of implementation that would influence the performance of the NSS model. The key issues addressed include:

- The selection of the optimisation algorithm: This study examined the fit of the NSS model under alternative optimisation algorithms. The choice between a constrained versus an unconstrained optimisation technique and its impact on the parameter stability is explored.
- The choice of starting parameter values: There has been little guidance in term structure literature on the ideal value of the starting parameters used to initialise an optimisation technique for yield curve estimation. This study puts in place alternative methods to determine the starting parameter values at the time of implementing the NSS model.
- The specification of the objective function: The results of the yield curve estimation are also compared by way of implementing various objective function specifications. At the time of minimising the sum of squared errors between the traded and model yields, this paper applies alternative weighting schemes in the objective function of the model by taking into account the size as well as the distribution of the errors.
- The Input Price Specification: Datasets are created based on entire days' trades, the last 3 trades and the last hour trades to examine the fit of the NSS model and the resulting parameter stability.

The yield curve was estimated under each specification of the objective function, the optimization, the starting parameter selection and the choice of the input price; to zero-in on a combination that would provide a best fit as well as parameter stability while implementing the NSS model.

On estimation of the ZCYC using the NSS model it was observed that a constrained optimization was better suited in the ZCYC yield curve estimation vis-à-vis an unconstrained optimization. It was further observed that the optimization solution is often

sensitive to the starting parameters selected while implementing the NSS model and was found that linking the starting parameter values of β_0 and β_1 to historical traded information could improve the estimation results.

When evaluating the effectiveness of the objective function, it was found that adopting a weighting scheme based on the distributional properties of the yield errors at the time of estimation helped in lowering the MAE on an average. The input price used was also found to be an important consideration in the ZCYC model estimation in terms of lowering the MAE. However, the resultant parameters were found to be stable irrespective of the price input used.

Based on the filtering criteria, the NSS model was estimated using a daily data sample of 2009-10 to 2019-20¹². The parameters associated with the long term and short term rates were found to be relatively stable during the period considered. The parameters were then used to estimate the historical interest rate term structure for the Indian government bond market. The results of the model YTM curve, the spot curve, the par curve and the forward curve were compared. The Mean Absolute Yield Error and Mean Absolute Price Error was then estimated to evaluate the effectiveness of the model.

This study has put in place a detailed framework for selection of appropriate filters to the data input and model specifications that is essential for implementing the NSS model while deriving the yield curve for the Indian government securities market. The study further computes the daily interest rates term structure for over a decade to arrive at security wise price/yield errors. The study attempts to provide insights into micro structure issues in the ZCYC estimation tested for IGBs. The analysis presented here serves as a useful resource for future research areas such as analyzing the relationship of the historical NSS parameter values with macro-economic variables, determinants of factors affecting price/yield errors, interest rate forecasting, valuation for Indian government bond markets etc.

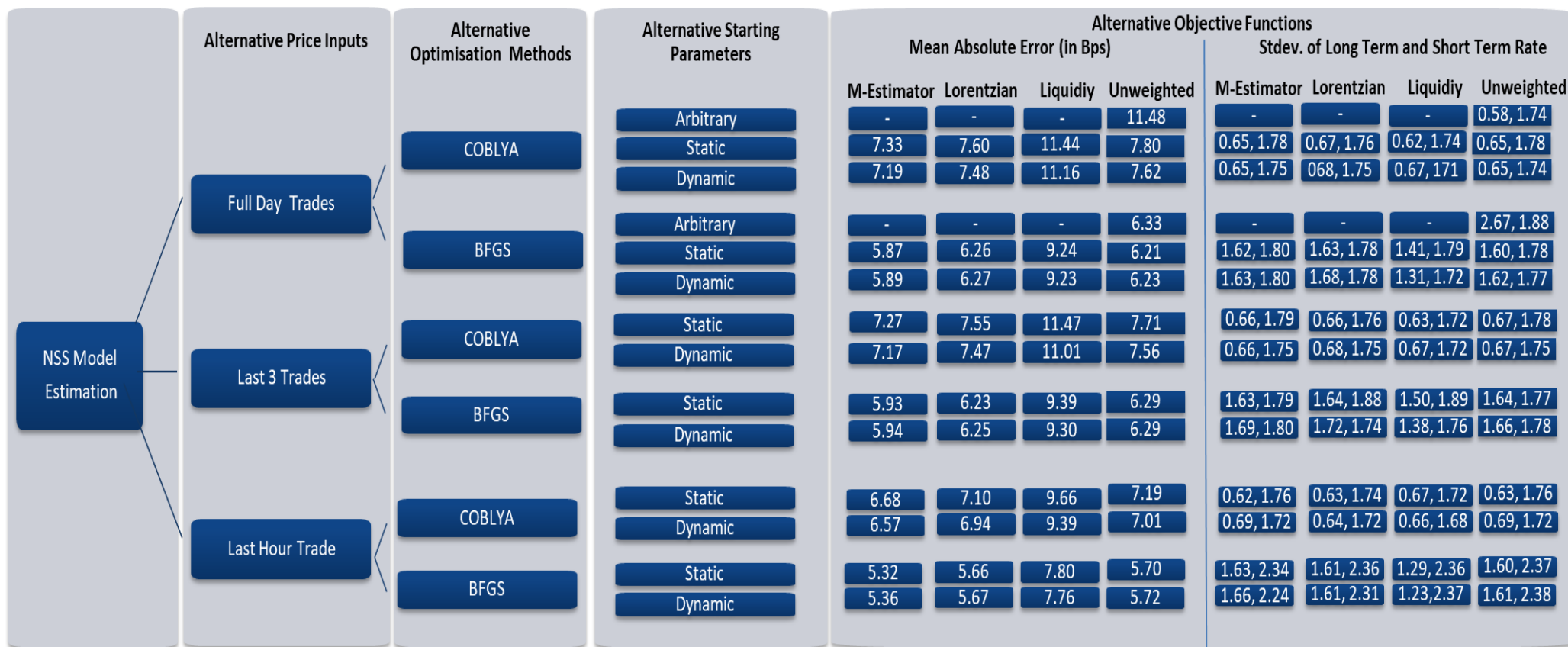
¹² The dataset is available on CCIL website along with this working paper.

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ANNEXURE 1

Flow Chart of Mean Absolute Yield Error (Bps) and Stdev. of NSS Model Parameters using Alternative Combinations (Jan 2009 to Mar 2020)



ANNEXURE 2

Comparison of MAPE under Alternative Model Specifications

Table A1 : Tenor-Wise Comparison of MAPE using Alternative Optimization Methods												
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: COBYLA Optimization												
0 To 5	0.14	0.15	0.18	0.19	0.11	0.17	0.16	0.10	0.08	0.18	0.12	0.15
5 To 10	0.96	0.69	0.67	0.51	0.81	0.62	0.55	0.54	0.56	0.60	0.53	0.66
10 To 15	0.96	0.61	0.58	0.41	0.51	0.72	0.45	0.76	0.85	0.68	0.49	0.65
15 To 20	2.55	1.92	2.07	1.48	0.65	1.23	0.95	0.81	0.84	1.36	0.55	1.30
20 To 40	2.65	2.25	2.85	2.42	0.96	2.25	1.74	1.38	1.29	1.93	0.94	1.82
Panel B: BFGS Optimization												
0 To 5	0.11	0.11	0.05	0.04	0.07	0.04	0.06	0.05	0.05	0.06	0.07	0.07
5 To 10	0.71	0.35	0.21	0.27	0.75	0.27	0.45	0.46	0.53	0.38	0.54	0.47
10 To 15	0.77	0.45	0.27	0.29	0.38	0.37	0.46	0.79	0.83	0.40	0.40	0.51
15 To 20	1.33	0.78	0.44	0.33	0.30	0.15	0.38	0.59	0.61	0.32	0.25	0.50
20 To 40	1.01	0.52	0.66	0.18	0.44	0.21	0.30	0.67	0.63	0.34	0.59	0.50

Table A2: Tenor-Wise Comparison of MAPE using COBYLA under Alternative Starting Parameter Estimates												
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Static Starting Parameters												
0 To 5	0.12	0.18	0.05	0.05	0.09	0.06	0.07	0.05	0.06	0.07	0.10	0.09
5 To 10	0.81	0.49	0.29	0.30	0.79	0.26	0.47	0.47	0.57	0.45	0.62	0.52
10 To 15	0.76	0.61	0.32	0.34	0.51	0.41	0.48	0.77	0.87	0.39	0.43	0.55
15 To 20	1.55	0.97	0.62	0.37	0.37	0.19	0.39	0.62	0.59	0.34	0.30	0.57
20 To 40	1.13	0.78	0.57	0.49	0.67	0.36	0.42	0.76	0.83	0.72	1.04	0.71
Panel B: Dynamic Starting Parameters												
0 To 5	0.13	0.14	0.05	0.05	0.09	0.07	0.06	0.06	0.07	0.10	0.10	0.08
5 To 10	0.83	0.44	0.28	0.29	0.75	0.26	0.48	0.46	0.56	0.42	0.63	0.51
10 To 15	0.76	0.56	0.34	0.33	0.51	0.42	0.48	0.78	0.86	0.43	0.41	0.55
15 To 20	1.51	0.82	0.64	0.37	0.42	0.22	0.35	0.61	0.59	0.34	0.32	0.56
20 To 40	1.19	0.69	0.50	0.42	0.78	0.30	0.35	0.71	0.74	0.57	1.06	0.66

Table A3: Tenor-Wise Comparison of MAPE using COBYLA under Alternative Weighting Schemes (with Dynamic Para.)												
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Unweighted Errors												
0 To 5	0.14	0.15	0.18	0.19	0.11	0.17	0.16	0.10	0.08	0.18	0.12	0.15
5 To 10	0.96	0.69	0.67	0.51	0.81	0.62	0.55	0.54	0.56	0.60	0.53	0.66
10 To 15	0.96	0.61	0.58	0.41	0.51	0.72	0.45	0.76	0.85	0.68	0.49	0.65
15 To 20	2.55	1.92	2.07	1.48	0.65	1.23	0.95	0.81	0.84	1.36	0.55	1.30
20 To 40	2.65	2.25	2.85	2.42	0.96	2.25	1.74	1.38	1.29	1.93	0.94	1.82
Panel B: Liquidity Based Weighted Errors												
0 To 5	0.21	0.15	0.07	0.08	0.11	0.05	0.10	0.10	0.10	0.10	0.09	0.11
5 To 10	0.74	0.42	0.19	0.33	0.93	0.33	0.64	0.62	0.97	0.50	0.94	0.62
10 To 15	1.37	0.44	0.35	0.32	0.97	0.61	0.83	1.11	1.68	0.69	1.42	0.91
15 To 20	3.35	1.97	1.32	0.89	1.23	0.55	1.04	1.56	1.86	1.12	1.44	1.47
20 To 40	3.45	1.97	1.22	1.04	1.23	0.66	1.02	2.03	2.87	1.54	1.72	1.69

Panel C: Lorengian Weighted Errors												
0 To 5	0.13	0.14	0.05	0.05	0.09	0.06	0.07	0.05	0.06	0.07	0.10	0.08
5 To 10	0.83	0.43	0.29	0.30	0.78	0.26	0.48	0.46	0.56	0.42	0.65	0.52
10 To 15	0.76	0.56	0.32	0.33	0.50	0.41	0.48	0.78	0.85	0.44	0.43	0.55
15 To 20	1.44	0.82	0.66	0.36	0.39	0.19	0.36	0.61	0.59	0.35	0.31	0.55
20 To 40	1.14	0.71	0.40	0.44	0.77	0.28	0.34	0.69	0.77	0.53	1.22	0.66
Panel D: Huber's M-estimates Based Weighted Errors												
0 To 5	0.12	0.14	0.05	0.04	0.09	0.07	0.07	0.05	0.06	0.06	0.09	0.08
5 To 10	0.84	0.40	0.27	0.29	0.75	0.25	0.46	0.47	0.55	0.42	0.62	0.50
10 To 15	0.72	0.56	0.33	0.36	0.51	0.38	0.45	0.77	0.81	0.44	0.39	0.53
15 To 20	1.28	0.80	0.60	0.28	0.27	0.18	0.33	0.57	0.59	0.27	0.29	0.49
20 To 40	0.95	0.69	0.42	0.32	0.69	0.30	0.34	0.67	0.69	0.55	1.06	0.60

Table A4: Tenor-Wise Comparison of MAPE using COBLYA under Alternative Data Filtering Samples with Dynamic Para. And M-estimator												
Period	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19	2019-20	Full Period
Panel A: Full Day's Trades												
0 To 5	0.12	0.14	0.05	0.04	0.09	0.07	0.07	0.05	0.06	0.06	0.09	0.08
5 To 10	0.84	0.40	0.27	0.29	0.75	0.25	0.46	0.47	0.55	0.42	0.62	0.50
10 To 15	0.72	0.56	0.33	0.36	0.51	0.38	0.45	0.77	0.81	0.44	0.39	0.53
15 To 20	1.28	0.80	0.60	0.28	0.27	0.18	0.33	0.57	0.59	0.27	0.29	0.49
20 To 40	0.95	0.69	0.42	0.32	0.69	0.30	0.34	0.67	0.69	0.55	1.06	0.60
Panel B: Last 3 Trades												
0 To 5	0.13	0.15	0.04	0.04	0.09	0.07	0.07	0.05	0.06	0.06	0.09	0.08
5 To 10	0.84	0.42	0.27	0.29	0.76	0.26	0.45	0.47	0.56	0.42	0.61	0.51
10 To 15	0.75	0.56	0.31	0.39	0.47	0.36	0.45	0.76	0.84	0.45	0.39	0.53
15 To 20	1.25	0.76	0.68	0.29	0.31	0.21	0.31	0.56	0.59	0.31	0.28	0.50
20 To 40	1.02	0.69	0.43	0.35	0.71	0.30	0.37	0.70	0.69	0.61	1.08	0.63
Panel C: Last Hour of Trading												
0 To 5	0.10	0.18	0.03	0.05	0.07	0.05	0.07	0.07	0.07	0.07	0.12	0.09
5 To 10	0.82	0.52	0.34	0.33	0.72	0.26	0.50	0.48	0.54	0.55	0.67	0.55
10 To 15	0.77	0.48	0.11	0.25	0.49	0.43	0.48	0.72	0.82	0.41	0.39	0.50
15 To 20	1.47	1.05	0.79	0.32	0.33	0.16	0.34	0.50	0.63	0.46	0.27	0.56
20 To 40	0.47	0.29	0.53	0.41	0.69	0.21	0.25	0.64	0.70	0.74	0.82	0.52